Electromagnetism and Special Relativity: Solution Set

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2 Problem 2: Time Dilation and Length Contraction

Consider me in my rest frame. I measure the time difference between an event at $(0,0,0,0)$ *and* $(t,0,0,0)$*. Now, boost these events to a rest frame moving at speed* v *with respect to me. Write the time difference this other observer sees between these two events as a relationship between the observed and proper time.*

The proper time between these two events is just $\Delta\tau = t - 0 = t$, as we are in the rest frame. In a boosted frame (say, in the x direction), these events are at $(0,0,0,0)$ and $(t\gamma, -vt\gamma, 0, 0)$ respectively, just from the Lorentz transformation. Therefore, the time difference as measured in that frame will be $\Delta t = \tau\gamma$, which is the formula for time dilation.

Next, I see an object moving towards me at speed v*. To measure it's length, I record simultaneous events at* $(0,0,0,0)$ *and* $(0,L,0,0)$*. Calculate where these events occur in the object's rest frame. The object can measure the location of it's two end points at different times, because it's not moving in it's rest frame. Thus, calculate the relationship between the proper length of the object and the length that I measure (proper length is the equivalent of proper time – it's the length of an object in the rest frame of that object).*

Length contraction is a little harder. Be careful with the Lorentz transformation here; the object is moving towards you, rather than away from you, which is the way we typically present the Lorentz transformation. Regardless, the two events transform to $(0,0,0,0)$ and $(Lv\gamma, L\gamma, 0, 0)$. Now, in my original frame, I needed to make the measurement of the two ends at the same coordinate time, as far as I saw it, because the object was moving. In the object's rest frame, we can indeed make a measurement of the length at two separate times, because the object isn't going anywhere. So, we see that the proper length $L_0 = L\gamma$, and a little rearrangement gives us $L(v) = L_0\sqrt{1 - v^2/c^2}$, which tells us how lengths contract depending on how fast you are moving.

3 Problem 3: Relativistic Doppler Shift

Consider a photon, of energy $E = hf$*, traveling in the* x *direction. Construct the four-momentum. What happens if we boost in the positive* x *direction? Construct the Lorentz transformation, and calculate the four-momentum in this frame. How does the frequency change as a function of velocity? Next, boost your original four-momentum in the* y *direction, and perform the same calculation. You should find that you have a Doppler shift, but your classical intuition shouldn't have expected one (it doesn't occur in sound waves, for example). This transverse relativistic Doppler effect is important in astronomy.*

The four-momentum for this photon is $p^\mu = (hf, hf, 0, 0)$. When we look at this four-momentum in a boosted frame in the x direction, this becomes $p^{\mu'} = (hf\gamma(1-v), hf\gamma(1-v), 0, 0)$. Thus, we have a relationship between the two observed frequencies of

$$f' = f\gamma(1-v) = f\sqrt{\frac{1-v}{1+v}} \tag{1}$$

from looking at the time component of the four-momentum.

Starting with our original four-momentum and boosting it in the y direction, the new four-momentum becomes $p^{\mu'} = (hf\gamma, hf, -hfv\gamma, 0)$. The relationship between the original and new frequencies is then

$$f' = f\gamma = \frac{f}{\sqrt{1-v^2}}. \tag{2}$$

4 Problem 4: Relativistic Addition of Velocity

Write down a velocity four-vector in the x direction. Now, boost this vector in the x direction by a different velocity. Calculate the resultant three-velocity from doing so. Now, instead of boosting in the x direction, boost in the y direction, and calculate the resultant three-velocity from this process. (Hint: Start with a zero velocity vector, and boost it to get the first velocity vector, using rapidity. When you perform the second boost, you can use hyperbolic trigonometric identities to perform this calculation straightforwardly.)

Our velocity vector is $v^\mu = \gamma_1(1, v_1, 0, 0)$. It is helpful to write this in terms of its rapidity, so $v^\mu = (\cosh\eta, \sinh\eta, 0, 0)$.

Now, let us boost this velocity in the x direction. Ie, we'd like to add a second velocity v_2 to v_1, and see what the resulting velocity must be. I'll use α for the rapidity associated with v_2, and use the boost formulas with rapidity.

$$\begin{aligned} v^{\mu\prime} &= (\cosh\eta\cosh\alpha + \sinh\eta\sinh\alpha, \sinh\eta\cosh\alpha + \cosh\eta\sinh\alpha, 0, 0) && (3)\\ &= (\cosh(\alpha+\eta), \sinh(\alpha+\eta), 0, 0) && (4)\end{aligned}$$

So, the new rapidity is $\alpha + \eta$. The new three-velocity will then be $v_3 = \tanh(\alpha+\eta)$. A little work yields the following, from expanding out the double angle formulae.

$$\begin{aligned} v_3 &= \frac{\sinh\eta\cosh\alpha + \cosh\eta\sinh\alpha}{\cosh\eta\cosh\alpha + \sinh\eta\sinh\alpha} && (5)\\ &= \frac{\gamma_1\gamma_2 v_1 + \gamma_1\gamma_2 v_2}{\gamma_1\gamma_2 + \gamma_1\gamma_2 v_1 v_2} && (6)\\ &= \frac{v_1+v_2}{1+v_1v_2} && (7)\end{aligned}$$

You can see why we like working with rapidities rather than velocities!

Now, let's take our original four-velocity, and boost it in the y direction by a rapidity α. The new four-velocity is

$$\begin{aligned} v^{\mu'} &= (\cosh\alpha\cosh\eta, \sinh\eta, \cosh\eta\sinh\alpha, 0) && (8)\\ &= (\gamma_1\gamma_2, \gamma_1 v_1, \gamma_1\gamma_2 v_2, 0) && (9)\\ &= \gamma_1\gamma_2\left(1, \frac{v_1}{\gamma_2}, v_2, 0\right). && (10)\end{aligned}$$

So, the new three-velocity is then $\mathbf{v} = v_1/\gamma_2\hat{x} + v_2\hat{y}$. Note that you get two different answers if you start with v_1 and boost it by v_2, or if you start with v_2 and boost it by v_1. See if you can figure out what's going on! (Answer in footnote)[1].

5 Problem 5: Longitudinal Electric Field

Starting with an electric field in the x direction, boost in the x direction. What happens to the electric field? What happens to the magnetic field? Explain what you have just derived.

[1] The issue here is most easily seen in the following. Start with the zero velocity, boost by velocity v_1, then boost again by velocity v_2. Compare to boosting in the reverse order. Boosts are not commutative, which should not be surprising, as rotations aren't commutative either. A product of two boosts is the same as a single boost plus a rotation, and so what's happened here is that the two different orders end up pointing in slightly different directions.

One can start here by writing the field strength tensor, and then boosting it with two Λ matrices. The original tensor should look like:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & 0 & 0 \\ E_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

If you did the problem this way, that's fine. I would just like to show you another way to do it, which will arrive at the same answer, but through a nifty use of the four-potential.

Consider $A^{\mu} = (-E_x x, 0, 0, 0)$. This yields the desired electric field, and is a little easier to Lorentz transform. Let's boost our frame of reference in the x direction. The four-potential in the new frame will be $A^{\mu'} = (-E_x \gamma^2 (x' + vt'), E_x v \gamma^2 (x' + vt'), 0, 0)$. We do need to remember to be careful to change the x coordinate to its x' equivalent in doing so; remember that this uses the inverse Lorentz transformation.

Now that we have our four-potential, we can just calculate the electric and magnetic field from it.

$$F'_{10} = E'_x = \partial_2 A_0 - \partial_0 A_2 = E_x \gamma^2 - E_x v^2 \gamma^2 = E_x \frac{1 - v^2}{1 - v^2} = E_x \tag{12}$$

$$F_{20} = E'_y = \partial_2 A_0 - \partial_0 A_2 = 0 \tag{13}$$

$$F_{30} = E'_z = \partial_3 A_0 - \partial_0 A_3 = 0 \tag{14}$$

$$F'_{12} = B'_x = \partial_1 A_2 - \partial_2 A_1 = 0 \tag{15}$$

$$F'_{31} = B'_y = \partial_3 A_1 - \partial_1 A_3 = 0 \tag{16}$$

$$F'_{23} = B'_z = \partial_2 A_3 - \partial_3 A_2 = 0 \tag{17}$$

Thus, our electric field is $\mathbf{E}' = E_x \hat{x}$, and our magnetic field is $\mathbf{B}' = 0$. So, if we only start with an electric field, if we boost in the direction of that electric field, the strength of the electric field doesn't change, and no magnetic field results from the transformation either.

6 Problem 6: Infinite Charged Plane

Consider an infinite charged plane with uniform surface charge density. Write down the electric field for this system, and then Lorentz boost it in the x direction. Describe the electric field, and explain if it is what you would expect, based on your original electric field and length contraction. What does the magnetic field look like? (Bonus question for later: Does this agree with Ampere's Law?)

Above the infinite plane, the electric field is $\mathbf{E} = \sigma/2\epsilon_0 \hat{z}$, while below, it's $\mathbf{E} = -\sigma/2\epsilon_0 \hat{z}$, and we have no magnetic field. Again, we could form the four-potential and transform that, but here, we'll just do it by transforming the field strength vector.

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E_z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_z & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

Here, $E_z = \pm\sigma/2\epsilon_0$ as appropriate. Boosting to a reference frame traveling in the x direction (and remembering to use the inverse transformation for the lowered indices), we need to contract with two Lorentz matrices.

$$F_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F_{\mu\nu} \tag{19}$$

Once you have worked out all the transformed components, you should find the following transformed field strength tensor.

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E_z\gamma \\ 0 & 0 & 0 & -E_z\gamma v \\ 0 & 0 & 0 & 0 \\ E_z\gamma & E_z\gamma v & 0 & 0 \end{pmatrix} \tag{20}$$

The electric field here looks just like the electric field from before, except that it's stronger by a factor of γ. This makes sense, as due to length contraction, the surface charge density is increased by a factor of γ, and so this checks out with Gauss' Law in the new frame.

The magnetic field here is in the y direction above the plane, and negative y direction beneath it. It's has constant strength $\sigma v\gamma/2\epsilon_0$. We can check with Ampere's Law to make sure this makes sense. For an infinite plane of current, Ampere's Law gives us $B = \mu_0 J/2$, where J is the current density, and the vector directions agree. The current density will just be $J = \sigma\gamma v$, where the factor of γ appears because of length contraction on the surface charge density. Using the relation $\mu_0 = 1/c^2\epsilon_0$, we find that we have agreement, up to factors of c, which we're not worried about anyway, and so our result agrees with Ampere's Law.

7 Problem 7: Lorentz Invariants

Calculate $F_{\mu\nu}F^{\mu\nu}$ in terms of the electric and magnetic fields. What is special about this quantity?

You can do this by brute force, or by being a little clever. Remember that diagonal elements are vanishing for $F_{\mu\nu}$. Using the three-dimensional epsilon tensor (see the Appendix to the notes), we can write $F_{ij} = \epsilon_{ijk}B_k$. I'm being sloppy with the spatial indices raised and lowered here, as the spatial metric is just the Kronecker delta. Note that $\epsilon_{ijk}\epsilon^{ijl} = 2\delta^l_k$.

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij} &(21)\\ &= 2F_{0i}F^{0i} + \epsilon_{ijk}B_k\epsilon^{ijl}B_l &(22)\\ &= -2E_iE_i + 2\delta^l_kB_kB_l &(23)\\ &= -2\mathbf{E}^2 + 2\mathbf{B}^2 &(24)\end{aligned}$$

This quantity is special because it is a *scalar* quantity, which is the same in all references frames.

8 Problem 8: Gauge Transformations

Recall that the electric potential has some freedom in where you choose the zero. When you move to a four-potential, the amount of freedom you have available increases. Show that the electric and magnetic fields are unchanged if you add a divergence to the electromagnetic potential, $A^\mu \to A^\mu + \partial^\mu f(x^\nu)$, where f is any differentiable function of spacetime. Calculate how this transformation (known as a gauge transformation) affects the electric potential and magnetic vector potential.

Consider the field strength tensor $F_{\mu\nu}$.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{25}$$

If we change $A^\mu \to A^\mu + \partial^\mu f(x^\nu)$, then this becomes

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu(A_\nu + \partial_\nu f) - \partial_\nu(A_\mu + \partial_\mu f) &(26)\\ &= \partial_\mu A_\nu + \partial_\mu\partial_\nu f - \partial_\nu A_\mu - \partial_\nu\partial_\mu f &(27)\\ &= \partial_\mu A_\nu - \partial_\nu A_\mu &(28)\end{aligned}$$

as partial derivatives commute. Thus, the field strength tensor is unchanged under this transformation.

The electric potential is usually $\phi = A^0$, and the magnetic vector potential is just A^i. Under this transformation, we have

$$\begin{aligned} \phi &\to \phi - \frac{\partial f}{\partial t} &(29)\\ \mathbf{A} &\to \mathbf{A} + \nabla f. &(30)\end{aligned}$$

Remember to include the -1 for raising the index on A^μ appropriately in the transformation for the potential.

9 Problem 9: Continuity Equation

Consider the equation $\partial_\mu F^{\nu\mu} = 4\pi J^\nu$. What happens if we differentiate again and contract with the free index? Write the resulting equation in terms of the three-dimensional charge density and current. This equation is known as the continuity equation, and tells you that charge must be conserved. As a bonus challenge, see if you can figure out how to derive conservation of charge from the equation.

Take the derivative, and see what happens.

$$\partial_\nu \partial_\mu F^{\nu\mu} = 4\pi \partial_\nu J^\nu \tag{31}$$

Now, $F^{\mu\nu}$ is antisymmetric, while the partial derivatives are symmetric, and so the left hand side must vanish. Therefore, we have

$$\partial_\mu J^\mu = 0. \tag{32}$$

If you write this out component by component, you'll get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{33}$$

which is none other than our old friend, the continuity equation.

To get to conservation of charge, integrate the continuity equation over some volume.

$$0 = \frac{\partial}{\partial t} \int_V \rho dV + \int_V \nabla \cdot \mathbf{j} dV \tag{34}$$

$$= \frac{d}{dt} Q_{enc} + \int_V \nabla \cdot \mathbf{j} dV \tag{35}$$

Using the divergence theorem on this second integral, we have

$$\frac{d}{dt} Q_{enc} = -\int_{\partial V} \mathbf{j} \cdot d\mathbf{A} \tag{36}$$

where the second integral is now an integral over the surface of the volume. Note that we changed from partial derivatives of time to total derivatives of time, because Q_{enc} only depends on time now. This tells us that the time rate of change of the charge enclosed in the volume is determined by the rate at which charge is leaving it. If we take our volume to go to infinite radius (or as big as necessary), such that no charge is leaving the sphere, then $dQ/dt = 0$, and charge is conserved.