Electromagnetism and Special Relativity: Four Beatles

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1 The Problem

One day, John set up a careful physics experiment. He constructed a constant electric field pointing upwards (in the z direction), and a constant magnetic field pointing sideways (in the y direction). Then, he found a little ball bearing, and added some positive charge to it, until the force from the electric field exactly balanced the gravitational field, and the ball bearing floated. The required condition, he noted, was $qE = mg$.

Then, he invited his friends Paul, Ringo and George to come and take a look. Paul was sitting nearby, and just had to look up to see the ball bearing floating. He agreed that $qE = mg$, and went back to work. Ringo was skateboarding, and travelled towards the ballbearing at a velocity v in the $-\hat{x}$ direction. He saw the ballbearing floating, but didn't understand why - in his frame of reference, the ballbearing had a velocity $v\hat{x}$, and so the magnetic field should have been pushing the ballbearing upwards. George was riding his bike from the other direction, and also saw the ballbearing floating, but he too was confused, because he saw the ballbearing moving towards him at a velocity $-v\hat{x}$, and so it should have been pushed down by the magnetic field.

Ringo and George argued with Paul for a bit, before John stepped into the fray, and declared what was really going on...

2 The Resolution

"In Paul's reference frame," said John, "he measures the following electromagnetic field strength tensor." He scribbles on a nearby blackboard.

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E_0 \\ 0 & 0 & 0 & -B_0 \\ 0 & 0 & 0 & 0 \\ E_0 & B_0 & 0 & 0 \end{pmatrix} \tag{1}$$

"I'll use E_0 and B_0 for the quantities in Paul's frame. Now, Paul also measures $qE_0 = mg$, so the ball bearing doesn't fall." He points to the floating shiny ball. "Now, everybody has to see the same thing, so it's no surprise that the ball is floating. It can't sink and rise at the same time. So something else has to change for George and Ringo to compensate."

"But what?" asks George.

"The electromagnetic field strength tensor!" John exclaims. "It has to transform to your new frame of reference! To get from Paul's reference frame to George's, we'd need to Lorentz boost in the x direction." He scribbles down the Lorentz transformation matrix.

$$\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

"We'll get to you in a minute, Ringo. Now, to transform the electromagnetic field strength tensor, we'll need the inverse transformation matrix, because the indices on $F_{\mu\nu}$ are lowered. So, we'll need this matrix." Again, he scribbles on the board.

$$\Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

"But how do we transform the thing?" asks Ringo.

"Simple, like this!" With a flourish, John constructs the transformed electromagnetic field strength tensor. The following equation appears on the board.

$$F_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}F_{\mu\nu} \tag{4}$$

"Ok, now for the hard part: calculating this thing..."

A few minutes pass, while the matrix calculation is performed. There are arguing over signs, but at the end, everybody agrees that the transformed tensor looks as follows.

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\gamma(E_0 + vB_0) \\ 0 & 0 & 0 & -\gamma(B_0 + vE_0) \\ 0 & 0 & 0 & 0 \\ \gamma(E_0 + vB_0) & \gamma(B_0 + vE_0) & 0 & 0 \end{pmatrix} \tag{5}$$

"Now what do we do?" asks Paul.

"Now it's time to calculate what George sees!" claims John. "Let's calculate the force!" Once again the chalk is wielded, and the following equation appears.

$$\mathbf{F}' = q\left(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}'\right) + \mathbf{F}'_g \tag{6}$$

"The Lorentz force law, plus the force due to gravity. Well, we can read off our electric and magnetic fields..."

$$\mathbf{E} = \gamma(E_0 + vB_0)\hat{z} \tag{7}$$
$$\mathbf{B} = \gamma(B_0 + vE_0)\hat{y} \tag{8}$$

"And $\mathbf{v}'$ is $-v\hat{x}$ for me!" chimes in George.

"Right, so we only need to look in the $\hat{z}$ direction. We have the following force in the $\hat{z}$ direction."

$$F_y = q(\gamma(E_0 + vB_0) - v\gamma(B_0 + vE_0)) + F'_g \tag{9}$$
$$= q(\gamma E_0 + \gamma v B_0 - v\gamma B_0 + \gamma v^2 E_0) + F'_g \tag{10}$$

"And the magnetic field parts cancel!" exclaims Ringo excitedly.

"Exactly!" replies John.

"But what about the gamma factor?" asks Paul, dubiously. He scratches another couple of lines beneath those two.

$$F_y = q\gamma E_0(1 + v^2) + F'_g \tag{11}$$
$$= \frac{qE_0}{\gamma} + F'_g \tag{12}$$

"See? We don't have an exact cancelation."

"Hmm," thinks John. "I think we need to look at the force carefully." He carefully constructs the following equation.

$$F^\mu = \frac{dp^\mu}{d\tau} = (0, 0, -mg, 0) \tag{13}$$

"In Paul's reference frame, this is what he sees as the acceleration due to gravity. Now, when we Lorentz transform this, we get..." He looks disheartened as he scratches the next equation onto the board, performing the Lorentz transformation in his head.

$$F^{\mu'} = \frac{dp^{\mu'}}{d\tau} = (0, 0, -mg, 0) \tag{14}$$

"... the same thing."

"That's ok," says Ringo. Because that's not the three-vector force that George would measure. That's the derivative with respect to the proper time of the ball bearing. We need to cast it into the time that George measures, because that's what the Lorentz force is doing too." He adds one more line to the calculation.

$$F^{\mu'} = \frac{dp^{\mu'}}{dt}\frac{dt}{d\tau} = \frac{dp^{\mu'}}{dt}\gamma = (0, 0, -mg, 0) \tag{15}$$

"We know $t = \gamma\tau$ from time dilation, and this," he says, circling $dp^{\mu'}/dt$, "is our three-vector force. So we get..."

$$\begin{aligned} F_y &= q\gamma E_0(1 + v^2) - \frac{mg}{\gamma} && (16)\\ &= \frac{1}{\gamma}(qE_0 - mg) && (17)\\ &= 0. && (18) \end{aligned}$$

The final dot makes a satisfying tap on the blackboard, and everyone is satisfied.

"So that's how it works..." mutters George. "Very good. But what about Ringo?"

"That's easy," says John. "Just let v become $-v$ wherever you see it, and you'll get Ringo's frame of reference."

Satisfied that they understood relativistic physics, the four friends left the office, and went on to record their Abbey Road album.