

The Thermodynamics and Fractional Statistics of the Spinor Bose Gas

Jolyon Bloomfield

A thesis submitted for the degree of
Bachelor of Philosophy with Honours in Physics at
The Australian National University

November, 2006



Declaration

This thesis is an account of research undertaken between February 2006 and November 2006 at The Department of Theoretical Physics, Research School of Physical Sciences and Engineering, The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any University.

Jolyon Keith Bloomfield
November, 2006

Acknowledgements

This year has been fairly intense, and there are a number of people who have helped me get through it who I would like to thank. I'd like to start with my supervisor Murray Batchelor, who despite conveniently forgetting to tell me just how difficult his project was, managed to get me through to the other side relatively unscathed. I'd like to especially thank Xi-Wen Guan, who was a fountain of knowledge whenever I had a technical question.

I would like to thank the Defence Science and Technology Organisation and the Australian Institute of Physics for awarding me the inaugural DSTO-AIP Scholarship, which has covered my expenses this year, allowing me to focus on this work.

I would not be here today if my parents hadn't encouraged me to move so far away from home, and the help that they gave in setting me up in Canberra is most appreciated. I'd also like to thank my siblings for letting me run away from them.

I'd like to particularly thank Colin Taylor of the Science Olympiad program, who was instrumental in convincing me that my career lay in physics.

Finally, I'd like to give special thanks to my girlfriend Ruth, who has helped me through this difficult year, and has stuck by me the whole way.

Abstract

We present a comprehensive analysis of the exact Bethe ansatz solution for the one-dimensional spinor Bose gas. The Bethe ansatz equations and the thermodynamic Bethe ansatz equations are derived, before investigating properties of these equations in limiting regimes. We explore the difference between spinless and spinor particles in the one-dimensional Bose gas. For the spinor Bose gas, we present a new approximation to the thermodynamic Bethe ansatz equations which allows them to be solved in the strong coupling regime. We derive the first order correction to the solutions of the thermodynamic Bethe ansatz equations in a low temperature regime, where spin effects are most evident. We then derive thermodynamic quantities for this model, adding a first order correction to the thermodynamics of the spinless system.

The subject of fractional exclusion statistics is presented, and its significance with regards to the spinless Bose gas is discussed. We also discuss its significance with regards to the spinor Bose gas, and find a regime in which non-mutual statistics may be applied to the spinor Bose gas to obtain expansions of thermodynamic quantities. These expressions are seen to improve upon previous analytical results for the model.

Contents

Declaration	i
Acknowledgements	iii
Abstract	v
Preface	xi
1 Introduction	1
1.1 Overview	1
1.1.1 Bose-Einstein Condensates	1
1.1.2 Experiments in One Dimension	1
1.1.3 The Spinor Bose Gas	2
1.2 Motivation	3
1.2.1 Fractional Exclusion Statistics	3
1.3 Thesis Plan	4
2 The Bethe Ansatz	5
2.1 The Spinless Boson with δ -function Interaction	5
2.2 The Spinor Boson with δ -function Interaction	6
2.2.1 The Hamiltonian	6
2.2.2 The Bethe Ansatz	7
2.2.3 The Bethe Ansatz Equations	7
2.2.4 Spin Rapidities	11
2.2.5 The Ground State	12
3 The Thermodynamic Bethe Ansatz	13
3.1 The Spinor Boson	13
3.2 The String Hypothesis	13
3.3 The Bethe Ansatz Equations	14
3.4 Logarithmic Form	16
3.5 Quantum Numbers	18
3.6 Integral Bethe Ansatz Equations	18
3.7 The Thermodynamic Bethe Ansatz	20
3.8 Alternate Form of the Thermodynamic Bethe Ansatz Equations	23
3.9 The Helmholtz Free Energy	24
3.10 Summary	26
4 Fractional Exclusion Statistics and the Thermodynamic Bethe Ansatz	29
4.1 Haldane Exclusion Statistics	29
4.2 Thermodynamics from Haldane's Principle	31
4.3 FES of the Spinless Boson	32

4.4	FES of the Spinor Boson	34
4.5	Summary	35
5	Limiting Regimes	37
5.1	$c = 0$	37
5.2	$c = \infty$	37
5.3	$T = 0$	39
	5.3.1 $M = 0$	40
	5.3.2 $M = 1$	41
	5.3.3 Derivations using arbitrary M	43
	5.3.4 The Fermionisation of Bosons	44
	5.3.5 Energies at $T = 0$	45
5.4	$T = \infty$	46
5.5	Summary	46
6	A New Approximation	49
6.1	Low T , Strong Coupling Regime	49
6.2	Solving the TBA Equations	50
6.3	Calculating the Free Energy	53
6.4	Comparisons to Other Regimes	57
	6.4.1 $c = \infty$	57
	6.4.2 $T = 0$	58
6.5	Explaining the Discrepancies	58
6.6	Summary	59
6.7	Further Work	59
7	Thermodynamics from Fractional Exclusion Statistics	61
7.1	Attempts at Incorporating the String Hypothesis	61
7.2	The Significance of the Statistical Interaction	61
7.3	A Special Case	62
8	Conclusions and Further Research	65
8.1	Conclusions	65
8.2	Further Research	66
A	Sommerfeld's Lemma	67
B	Glossary	69
	Bibliography	71

List of Figures

1.1	Comparison between experiment and theory for the 1D Bose gas	2
1.2	Illustration of the process of fermionisation	3
5.1	Density functions at $T = 0$ for varying c	42
5.2	Density functions for $M = 0$ and $M = 1$	43
5.3	Ground state energy for $M = 0$ and $M/N = 1/2$ as a function of c	46
6.1	Comparison between approximate and correct integrand for η	51
6.2	Comparison between approximation and original function for Equation 6.11	52

Preface

Throughout this thesis, a number of conventions have been used in mathematical equations for brevity. We use the conventions

$$\begin{aligned}\hbar &= 1 \\ 2m &= 1 \\ k_B &= 1\end{aligned}$$

throughout, where \hbar is Plank's constant divided by 2π , m is particle mass, and k_B is Boltzmann's constant.

To indicate the Fourier transform of a function, we place a hat over the function. For example, $\hat{\sigma}$.

We have used $\mathcal{O}(x)$ to denote big-O notation, indicating terms of order x .

An approximation known as "Sommerfeld's Lemma" is sometimes used. Details on Sommerfeld's lemma may be found in Appendix A.

Introduction

1.1 Overview

Recent developments in experimental techniques have seen the production of a variety of quantum structures, including quantum gases, quantum dots and wires, Bose-Einstein condensates (BECs), and superconducting metallic nanograins. The ability to obtain nanokelvin regimes, highlighted in the production of BECs, has allowed theories of novel quantum structures and phase transitions to be tested experimentally for the first time.

Fundamental quantum systems involve the interaction between bosonic and/or fermionic particles. At low temperatures, subtle effects become evident, such as those of spin interactions, because thermal fluctuations have been suppressed. Furthermore, statistical effects such as the Pauli exclusion principle for fermions become highlighted. In this regard, fermions and bosons behave very differently at low temperatures. Fermions are particles with half-integral spin which obey the Pauli exclusion principle, while bosons are particles with integral spin which do not. A curious property of quantum mechanics is that when two fermions are combined, they form a boson, and so depending on the nucleus and the number of electrons an atom has, it may be either a boson or a fermion. Using this principle, gases of alkaline metals of specific nuclei have been made to condense at very low temperatures to form a Bose-Einstein condensate, where large numbers of particles collapse into the quantum ground state. A consequence of this phenomenon is that quantum effects are amplified, to the extent that some may be observed macroscopically.

1.1.1 Bose-Einstein Condensates

Bose-Einstein condensates form when the de Broglie wavelength of particles becomes so large that the wavefunctions for particles overlap significantly. This is achieved by cooling a gas of atoms to a very low temperature, often through the use of laser cooling and evaporative cooling, although a variety of new ways to trap and cool atoms have been developed. The first gaseous BEC was observed in 1995, for which the Nobel Prize in Physics was awarded in 2001 [1]. The technique used to do so, a method known as “laser cooling”, which cools and traps atoms with laser light, was awarded the Nobel Prize in 1997 [2]. Although remarkable in itself, BECs and techniques associated with them have allowed experimentalists to probe nanokelvin regimes to investigate a variety of new quantum systems.

1.1.2 Experiments in One Dimension

Building on the success of BECs, it is now possible to create effectively one-dimensional traps for quantum gases, with a tunable interaction strength [3, 4, 5]. Common techniques

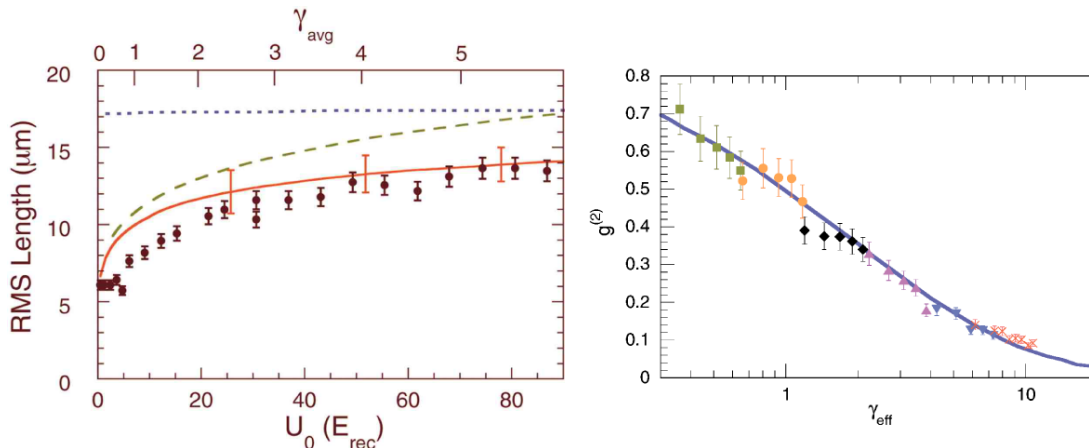


Figure 1.1: Comparison between experiment and theory for the 1D Bose gas. The left panel shows the rms full length of the 1D atom cloud *vs.* the transverse confinement [5]. The circles represent the measured values. The solid line is the exact result, and the dotted line is the fermionic limit. Error bars on the theory curve reflect uncertainties in experimental parameters. The right panel shows the local pair correlation function *vs.* effective interaction strength of the 1D Bose gas [7]. Again the solid line is obtained from the exactly solved (integrable) model [6].

for the production of one-dimensional traps include optical lattices and optical waveguides.

Notable experiments include the measurement of momentum distribution profiles for one-dimensional systems [3, 4, 5]. Figure 1.1, left panel, plots the results of Weiss *et al.*, who measured the internal energy of the system as a function of system scale and temperature (the scale of the system is shown here), for varying interaction strength. It was found that the results fitted the predictions of the one-dimensional Bose gas (also called a Tonks-Girardeau gas for strong coupling), agreeing with Lieb and Liniger’s seminal paper on the subject [6]. The trap was comprised of an array of ‘tubes’ in which atoms were restricted to movement in one dimension only.

Another striking experiment, also by Weiss *et al.* [7], measures the photoassociation rates in one-dimensional Bose gases of ^{87}Rb atoms to determine the local pair correlation function over a range of interaction strengths (see Figure 1.1, right panel). The local pair correlation function provides an indication of atomic wavefunction overlap, and so this experiment provides a direct observation of the fermionisation of bosons with increasing interaction strength. Fermionisation is a process by which bosons with a repulsive interaction start to behave like fermions as the interaction strength is increased. Figure 1.2 provides a qualitative illustration of the process.

1.1.3 The Spinor Bose Gas

Like the quantum Bose gas, it is now also possible to trap multicomponent gases through manipulation of hyperfine Zeeman states, creating an effective spinor Bose gas. The structure of the spinor Bose gas is much richer than that for the quantum Bose gas, as the spin degree of freedom can lead to spin oscillations, causing spin waves. The observation of the dynamics of spin waves and the effect of spin-states on trapped spinor Bose gases provide further exciting opportunities for studying magnetism and novel quantum phase transitions [8, 9, 10]. To begin with, the ground state has ferromagnetic order, which is

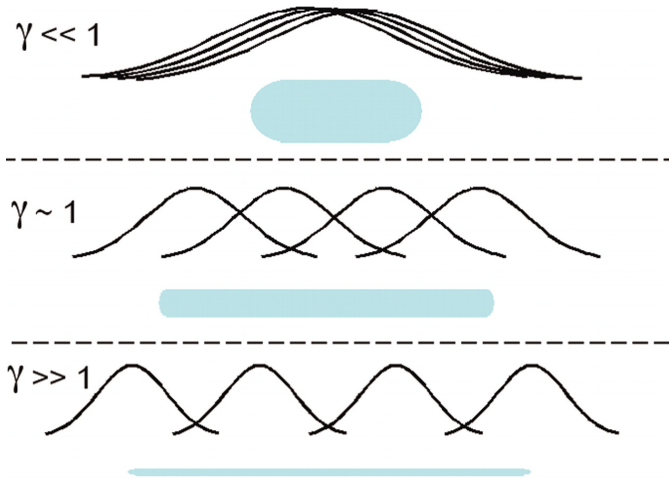


Figure 1.2: Illustration of atomic distributions and wavefunctions in one dimension. For small interaction strength (low γ), the wavefunctions have significant overlap, and the particle distribution is concentrated in a small region. As the interaction strength increases, the particles begin to repel each other more, so there is less overlap between wavefunctions, and the particle distribution becomes broader. Figure taken from Ref. [5].

significantly different to the antiferromagnetic behaviour found in Fermi gases. Far less work has been done to date on the spinor Bose gases [11, 12, 13, 14, 15] compared to the spinless case.

1.2 Motivation

One of the features of one-dimensional models is that they are often exactly solvable quantum many-body problems, such as the 1D Hubbard model [16], the BCS model [17, 18], and the Heisenberg chain [19]. Sutherland has referred to these as “Beautiful Models” [20], stating that while such models are too mathematical to be physics, and yet not rigorous enough to be mathematics, they provide an unsurpassed insight into the realm of many body physics. A curious consequence of one-dimensionality is that particle interchange may only occur when two particles pass through each other, feeling the full scope of particle-particle interactions. Given the experimental success of the Lieb-Liniger model for one-dimensional spinless bosons [6], analytical results for more complex models are now of particular interest, as the theoretical results now may be experimentally tested.

Our aim in this thesis is to explore the spinor Bose gas model, which presents an insight into the effect of an internal degree of freedom possessing an $SU(2)$ symmetry on a many-body system. We investigate the thermodynamics of this model, with particular attention devoted to the strong coupling regime at low temperatures, in which the fermionisation of bosons is evident.

1.2.1 Fractional Exclusion Statistics

On a more theoretical note is the classification of a system by the statistics that its particles obey. Well known quantum statistics are the Bose-Einstein statistics and Fermi-Dirac statistics. Yet some systems belong to neither of these, and are attributed to “fractional

exclusion statistics". It has been shown [21] that the spinless Bose gas in one dimension obeys Haldane exclusion statistics [22], and it is thus of theoretical interest to understand what manner of statistics the spinor Bose gas obeys, if any. We examine this question, and look into the possibility of deriving thermodynamic results from this approach.

1.3 Thesis Plan

In Chapter 2, we present the Bethe ansatz method, which is the method used to derive exact solutions for key one-dimensional models. We review results from the quantum Bose gas, before deriving the Bethe ansatz equations for the spinor Bose gas in detail. The thermodynamics of the spinor Bose gas are derived in Chapter 3, where we obtain the thermodynamic Bethe ansatz equations.

The concept of fractional exclusion statistics is discussed in Chapter 4. We present the motivation for Haldane exclusion statistics, and derive thermodynamic results for a system which obeys such statistics. We review Bernard and Wu's presentation of how the spinless Bose gas obeys Haldane exclusion statistics, before considering how such a result may be applied to the spinor Bose gas.

Having developed the requisite background theory, we present our own derivations of solutions to the thermodynamic Bethe ansatz equations in limiting regimes in Chapter 5, in order to understand the behaviour of the model at strong and weak coupling, as well as at high and low temperatures.

In Chapter 6, we present a new approximation to the thermodynamic Bethe ansatz equations at low temperature in the strong coupling regime, deriving a first order correction to the thermodynamics for the spinless Bose gas.

We return to fractional exclusion statistics in Chapter 7, and discuss why all attempts at deriving a statistical interaction for the spinor Bose gas have failed. However, we have found a special situation in which a statistical interaction may be approximated, and apply the work of Isakov *et al.* [23] to obtain thermodynamic information.

Finally, we conclude this work and describe possibilities for further investigation in Chapter 8.

We have drawn together a list of terms with which the reader may be unfamiliar, and presented a brief explanation of these in a glossary.

The Bethe Ansatz

The Bethe ansatz was introduced in 1931 by Hans Bethe [24] as a proposed method to obtain the eigenspectrum of the one-dimensional Heisenberg ferromagnet. The principal assumption of the method is the Bethe-Hulthén hypothesis [25], which states that in the scattering of two particles, the outgoing states yield only reflected waves and no diffracted waves, which has been shown to be equivalent to being able to divide coordinate space into a finite number of regions. If this is the case, then the wavefunction may be assumed to be a finite sum of travelling waves of different amplitudes. Importantly, these eigenfunctions are complete, and account for all possible states. This method has proven to be invaluable in finding analytical solutions for a number of one-dimensional models. We present the results from using the Bethe ansatz for the spinless Bose gas, before deriving the Bethe ansatz equations for the spinor Bose gas.

2.1 The Spinless Boson with δ -function Interaction

In 1963, Lieb and Liniger [6] applied the Bethe ansatz to the problem of one-dimensional bosons with a δ -function interaction. They derived the general wavefunction and Bethe ansatz equations, before extensively analysing the ground state.

The Hamiltonian of this system is

$$\mathcal{H} = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j) \quad (2.1)$$

where N is the number of particles in the system, c is the interaction strength between particles, and $\{x_i\}$ refers to the positions of the particles. Many models use this form of interaction; notably, the Gross-Pitaevskii equation starts with this interaction before applying a mean-field approximation. The Bethe ansatz is a trial wavefunction for this Hamiltonian

$$\psi(x_1, x_2, \dots, x_N) = \sum_P A(P) \exp(i \sum_l^N k_{P_l} x_l) \quad (2.2)$$

where P denotes the sum over all permutations of $(1 \ 2 \ 3 \ \dots \ N)$, and the parameters $\{k_i\}$ are pseudomomenta. Note that the form of the ansatz is a sum over all possible permutations of plane waves. The momentum and energy operators reveal

$$\begin{aligned}
K &= \sum_{i=1}^N k_i \\
E &= \sum_{i=1}^N k_i^2
\end{aligned}
\tag{2.3}$$

From considering the periodic boundary conditions, particle exchange symmetries, and the discontinuity in derivative of the wavefunction where two particles overlap, relationships between the pseudomomenta can be found. These relationships generally result in N transcendental equations, known as the Bethe ansatz equations (BAE). Lieb and Liniger found these equations to be

$$e^{ik_i L} = \prod_{j \neq i}^N \frac{k_i - k_j + ic}{k_i - k_j - ic}, \quad i = 1, 2, \dots, N
\tag{2.4}$$

where it is enforced that all $\{k_i\}$ are different, else the wavefunction identically vanishes. L is the length of the system.

As discussed in Chapter 1, the Lieb-Liniger gas has been experimentally realised, with results confirming predictions from the model.

2.2 The Spinor Boson with δ -function Interaction

Following on from the predictive success of the Lieb-Liniger model, a similar method may be employed to model the spinor Bose gas in one dimension. The spinor Bose gas is a system whose wavefunction is symmetrical upon particle exchange, and whose particles have an internal degree of freedom with $SU(2)$ symmetry. This is akin to a boson with a pseudo-spin structure that has the usual raising and lowering operators. Note that the $SU(2)$ group allows for an arbitrary maximum spin, but for spinor bosons, this maximum spin is $\frac{1}{2}$. This $SU(2)$ symmetry may also be generalised to higher order $SU(N)$ symmetry.

Although bosons usually have integer spin, we may construct situations in which particles are effectively constrained to spin $\frac{1}{2}$. We take, for example, an atom with angular momentum $F = I + J$, where I is the nuclear spin and J is the electron angular momentum. If $J = 1/2$, $I = 7/2$ in the ground state (for example, caesium), then possible F values are 3 and 4, which correspond to two separate hyperfine Zeeman states. When atoms are magnetically confined, they will adopt “weak field” seeking states to minimise their energy. Such a state will have a particular angular momentum and projection $|F, m_F\rangle$. It is possible to magnetically confine more than one such state together, as is the case for ^{87}Rb , where projections of the singlet and triplet states can coexist. This can lead to a gas of bosonic atoms with two spin states forming a pseudo-spin doublet; we may denote the two states as $|\uparrow\rangle$ and $|\downarrow\rangle$. These two states will have different scattering energies. In this way, a spinor pseudo-structure may be formed.

As the rest of this thesis deals with the Bethe ansatz equations for the spinor boson, they are derived here in detail. We follow the approach set out in Oelkers *et al.* [26, appendices].

2.2.1 The Hamiltonian

We use a similar Hamiltonian to Lieb and Liniger, adding in a Zeeman-type term that distinguishes between spin- \uparrow and spin- \downarrow particles. Ω is the coupling strength for a Rabi

interaction, and M is the number of particles with spin down, to a maximum of $N/2$. Thus

$$\begin{aligned}\mathcal{H} &= -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j) - 2\Omega \sum_{i=1}^N S_i^z \\ &= -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j) - \Omega(N - 2M)\end{aligned}\quad (2.5)$$

Here we introduce the pseudospin operators $S^z |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle$, $S^z |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle$. Note that c , which represents the interaction strength, must be positive for the thermodynamic limit to exist. This means that the interaction is of a repulsive nature.

2.2.2 The Bethe Ansatz

Our wavefunction is a function of the position of all particles, as well as a function of the spin of each particle. We write our ansatz as the sum over all travelling waves, with amplitudes dependent upon the order of the particles in one dimension, the spin states, and the permutation being summed.

$$\begin{aligned}\Psi(x_1, \dots, x_N, \sigma_1, \dots, \sigma_N) &= \sum_P A_{\sigma_1, \dots, \sigma_N}(P_1, \dots, P_N | Q_1, \dots, Q_N) \\ &\quad \times \exp(i(k_{P_1} x_{Q_1} + \dots + k_{P_N} x_{Q_N}))\end{aligned}\quad (2.6)$$

The sum is over all $N!$ permutations P of $(1 \ 2 \ 3 \ \dots \ N)$, and Q is the permutation (Q_1, Q_2, \dots, Q_N) of $(1 \ 2 \ 3 \ \dots \ N)$ such that $0 \leq x_{Q_1} < \dots < x_{Q_N} \leq L$. $\{k_i\}$ are N pseudomomentum values, to which restrictions will be derived. They must all be different, or the wavefunction becomes degenerate. Due to the nature of this piecewise definition based upon the order of the particles, the pseudomomenta are not true wavevectors. However, the sum of all pseudomomenta, which is the total momentum of the system, is a true wavevector.

The momentum and kinetic energy of the system

$$\begin{aligned}K &= \sum_{i=1}^N k_i \\ E &= \sum_{i=1}^N k_i^2\end{aligned}\quad (2.7)$$

are found by applying the momentum and energy operators.

2.2.3 The Bethe Ansatz Equations

To derive the Bethe ansatz equations, i.e., to derive the energy eigenspectrum of the model, we apply three conditions which the amplitudes obey:

1. Symmetry
2. Continuity, and discontinuity of the derivative
3. Periodic boundary conditions

Symmetry

Let us start by applying boson symmetry to the wavefunction: $\Psi \rightarrow \Psi$ upon interchanging $x_i, \sigma_i \leftrightarrow x_j, \sigma_j$:

$$A_{\sigma_i \sigma_j}(P_1, \dots, P_N | Q_a, Q_b) = A_{\sigma_j \sigma_i}(P_1, \dots, P_N | Q_b, Q_a)$$

A change in notation is introduced here for brevity, where Q_a and Q_b denote the positions of x_i and x_j . Similarly, let $i = P_a$, and $j = P_b$. Let us also introduce the operator $[T^{ij}]_{\sigma_1 \dots \sigma_N}^{\sigma'_1 \dots \sigma'_N} = \delta_{\sigma_i, \sigma'_j} \delta_{\sigma_j, \sigma'_i} \prod_{r \neq i, j} \delta_{\sigma_r, \sigma'_r}$, which obeys

$$A_{\sigma_1 \dots \sigma_N}(P | Q_a, Q_b) = [T^{ij}]_{\sigma_1 \dots \sigma_N}^{\sigma'_1 \dots \sigma'_N} A_{\sigma'_1 \dots \sigma'_N}(P | Q_b, Q_a)$$

The Einstein summation convention is used. Note that $T^{ij} = \mathcal{P}^{ij}$, where \mathcal{P} is the permutation operator.

Continuity

The wavefunction must be continuous at the intersection between two of the piecewise components. Thus we have $\Psi(\vec{x}, \vec{\sigma})_{x_i > x_j, x_i \rightarrow x_j} = \Psi(\vec{x}, \vec{\sigma})_{x_j > x_i, x_i \rightarrow x_j}$. Then in Ψ , there are pairs of permutations which have the same exponentials at $x_i = x_j$, leading to

$$\begin{aligned} & A_{\vec{\sigma}}(P_i, P_j | Q_i, Q_j) \exp(i(x_{Q_i} k_{P_i} + x_{Q_j} k_{P_j})) \\ & + A_{\vec{\sigma}}(P_j, P_i | Q_i, Q_j) \exp(i(x_{Q_i} k_{P_j} + x_{Q_j} k_{P_i})) = \\ & A_{\vec{\sigma}}(P_i, P_j | Q_j, Q_i) \exp(i(x_{Q_j} k_{P_i} + x_{Q_i} k_{P_j})) \\ & + A_{\vec{\sigma}}(P_j, P_i | Q_j, Q_i) \exp(i(x_{Q_j} k_{P_j} + x_{Q_i} k_{P_i})) \end{aligned}$$

Realising that x_{Q_i} and x_{Q_j} are identical at the intersection, the exponentials cancel, leaving

$$\begin{aligned} & A_{\vec{\sigma}}(P_i, P_j | Q_i, Q_j) + A_{\vec{\sigma}}(P_j, P_i | Q_i, Q_j) = \\ & A_{\vec{\sigma}}(P_i, P_j | Q_j, Q_i) + A_{\vec{\sigma}}(P_j, P_i | Q_j, Q_i) \end{aligned}$$

δ -interaction

If the Hamiltonian is integrated with respect to $d(x_i - x_j)$ from $-\epsilon$ to ϵ , it is readily found by changing to centre-of-mass coordinates and letting ϵ approach zero that

$$\left. \frac{\partial \Psi}{\partial y} \right|_{y=0^+} - \left. \frac{\partial \Psi}{\partial y} \right|_{y=0^-} = c \Psi|_{y=0}$$

Here, y is the $(x_i - x_j)$ coordinate. When this formula is applied to the ansatz (equation 2.6), we find that

$$\begin{aligned} & i(k_{P_b} - k_{P_a}) [A_{\vec{\sigma}}(P_a, P_b | Q_a, Q_b) - A_{\vec{\sigma}}(P_b, P_a | Q_b, Q_a)] \\ & = c [A_{\vec{\sigma}}(P_a, P_b | Q_a, Q_b) + A_{\vec{\sigma}}(P_b, P_a | Q_a, Q_b)] \end{aligned}$$

Using the T^{ij} operator defined above, we can rearrange these equations into

$$A_{\sigma_1 \dots \sigma_N}(P_a, P_b | Q_a, Q_b) = \underbrace{\left[\frac{i(k_{P_b} - k_{P_a})T^{ij} + c\text{Id}}{i(k_{P_b} - k_{P_a}) - c} \right]_{\sigma_1 \dots \sigma_N}^{\sigma'_1 \dots \sigma'_N}}_{\sigma_1 \dots \sigma_N} A_{\sigma'_1 \dots \sigma'_N}(P_b, P_a | Q_a, Q_b)$$

We use Id to denote identity operator. Define the operator $Y^{ij}(k_{P_b} - k_{P_a})$ to be the term in the brace. Because this relationship is derived from neighbouring particles, it is valid only for when x_i and x_j are neighbours, or equivalently, when Q_a and Q_b are neighbours. Finally, define $[X^{ij}(u)]_{\sigma_1 \dots \sigma_N}^{\sigma'_1 \dots \sigma'_N} = [Y^{ij}(u)]_{\sigma'_1 \dots \sigma'_N}^{\sigma_1 \dots \sigma_N} [T^{ij}(u)]_{\sigma_1 \dots \sigma_N}^{\sigma'_1 \dots \sigma'_N}$. Then we can write

$$A_{\sigma_1 \dots \sigma_N}(P_a, P_b | Q_a, Q_b) = [X^{ij}(u)]_{\sigma_1 \dots \sigma_N}^{\sigma'_1 \dots \sigma'_N} A_{\sigma'_1 \dots \sigma'_N}(P_b, P_a | Q_b, Q_a) \quad (2.8)$$

This is particularly useful as it allows us to swap two P and Q values.

Periodic Boundary Conditions

Enforcing periodic boundary conditions provides the next step in deriving the Bethe ansatz equations. Two points in one-dimensional space, call them $x = 0$ and $x = L$, are considered to be the same point. Thus, we require that $\Psi(x_1, x_2, \dots, x_i = 0, \dots, x_N) = \Psi(x_1, x_2, \dots, x_i = L, \dots, x_N)$. Without loss of generality, let $i = Q_1$. Then when $x_i = 0$, Q_1 will be in the leftmost position, but when $x_i = L$, Q_1 will be in the rightmost position. In full, we have

$$\sum_P A_{\sigma_1, \dots, \sigma_N}(P_1, \dots, P_N | Q_1, \dots, Q_N) e^{i(k_{P_2} x_{Q_2} + \dots + k_{P_N} x_{Q_N})} = \sum_P A_{\sigma_1, \dots, \sigma_N}(P_1, \dots, P_N | Q_2, \dots, Q_N, Q_1) e^{i(k_{P_1} x_{Q_2} + \dots + k_{P_{N-1}} x_{Q_N} + k_{P_N} L)}$$

Equating the exponentials on each side with regards to $\{x_i\}$, we arrive at the condition

$$\begin{aligned} & A_{\sigma_1, \dots, \sigma_N}(P_1, \dots, P_N | Q_1, \dots, Q_N) \\ &= e^{ik_{P_1} L} A_{\sigma_1, \dots, \sigma_N}(P_2, \dots, P_N, P_1 | Q_2, \dots, Q_N, Q_1) \end{aligned} \quad (2.9)$$

The Yang-Baxter Relations

Equation 2.9 provides a relationship between two different amplitudes A . Through the use of the X operator, defined in equation 2.8, it is possible to rewrite equation 2.9 in terms of one A only. This is done by permuting P_1 and Q_1 through from one side to the other.

There are many different paths which one can take to obtain this result, and it is important that all paths are equivalent. It is well known that operators which obey this condition satisfy the celebrated Yang-Baxter relations, which are

$$\begin{aligned} X^{ab}(u)X^{cd}(v) &= X^{cd}(v)X^{ab}(u) \quad \text{for different } a, b, c, d \\ X^{ab}(u)X^{ac}(u+v)X^{bc}(v) &= X^{bc}(v)X^{ac}(u+v)X^{ab}(u) \\ X^{ab}(u)X^{ba}(-u) &= 1 \end{aligned} \quad (2.10)$$

Ignoring the σ subscripts for now, $X^{ij}(u)$ may be written as

$$\begin{aligned} X^{ij}(u) &= Y^{ij}(u)T^{ij}(u) \\ &= \frac{i u T^{ij} + c \text{Id}}{i u - c} T^{ij} \\ &= \frac{i u \text{Id} + c T^{ij}}{i u - c} \end{aligned} \quad (2.11)$$

In this notation, it is easily seen that X^{ij} obeys the Yang-Baxter relations. Thus, all paths from a common start to a common end using multiple X^{ij} operations are equivalent.

We begin

$$\begin{aligned} &e^{i k_{P_1} L} A_{\sigma_1, \dots, \sigma_N}(P_2, \dots, P_N, P_1 | Q_2, \dots, Q_N, Q_1) \\ &= A_{\sigma_1, \dots, \sigma_N}(P_1, \dots, P_N | Q_1, \dots, Q_N) \\ &= X^{P_2 P_1}(k_{P_2} - k_{P_1})_{\vec{\sigma}}^{\vec{\sigma}'} A_{\sigma'_1, \dots, \sigma'_N}(P_2, P_1, P_3, \dots, P_N | Q_2, Q_1, Q_3, \dots, Q_N) \\ &= X^{P_2 P_1}(k_{P_2} - k_{P_1})_{\vec{\sigma}}^{\vec{\sigma}'} X^{P_3 P_1}(k_{P_3} - k_{P_2})_{\vec{\sigma}'}^{\vec{\sigma}''} \\ &\quad A_{\sigma''_1, \dots, \sigma''_N}(P_2, P_3, P_1, P_4, \dots, P_N | Q_2, Q_3, Q_1, Q_4, \dots, Q_N) \\ &\vdots \\ &= X^{P_2 P_1}(k_{P_2} - k_{P_1})_{\vec{\sigma}}^{\vec{\sigma}'} \dots X^{P_N P_1}(k_{P_N} - k_{P_1})_{\vec{\sigma}^{N-2}}^{\vec{\sigma}^{N-1}} \\ &\quad A_{\sigma_1^{N-1}, \dots, \sigma_N^{N-1}}(P_2, \dots, P_N, P_1 | Q_2, \dots, Q_N, Q_1) \end{aligned} \quad (2.12)$$

Thus, we have arrived at an eigenvalue equation for A . To find the eigenvalue of the product of X operators, we make the transformation

$$\begin{aligned} X^{ij}(u) &= \frac{i u \text{Id} + c T^{ij}}{i u - c} \\ &= \frac{i u + c}{i u - c} \cdot \frac{i u \text{Id} + c T^{ij}}{i u + c} \\ &= \frac{u - i c}{u + i c} R^{ij} \left(\frac{i u}{c} \right) \end{aligned} \quad (2.13)$$

where $R^{ij}(u) = \frac{u \text{Id} + P^{ij}}{1 + u}$. For ease of writing, we also make the transformation $P_2 = 1, P_3 = 2, \dots, P_N = N$, skipping i , to which we give $P_1 = i$. We do the same for Q_n , and adjust the indices accordingly. It is understood that $1 \leq i \leq N$, covering all possibilities. We also suppress the spin indices, as they remain unchanged, and are dealt with by Sutherland (see below). We then have

$$\begin{aligned} &X^{1,i}(k_1 - k_i) \dots X^{i-1,i}(k_{i-1} - k_i) X^{i+1,i}(k_{i+1} - k_i) \dots X^{N,i}(k_N - k_i) \\ &= \left[\prod_{j \neq i}^N \frac{k_i - k_j + i c}{k_i - k_j - i c} \right] R^{1,i}(k_1 - k_i) \dots R^{i-1,i}(k_{i-1} - k_i) \\ &\quad R^{i+1,i}(k_{i+1} - k_i) \dots R^{N,i}(k_N - k_i) \end{aligned} \quad (2.14)$$

Thus the problem is reduced to finding the eigenvalue of the transfer matrix associated with R^{ij} . This was first accomplished by Sutherland [27] in 1967, who discovered that solving the system produced another set of equations of the same form, with the representation reduced by one dimension. The first level Bethe ansatz equations are then found

by equating the eigenvalue to the exponential previously found.

$$e^{ik_i L} = \prod_{j \neq i}^N \frac{k_i - k_j + ic}{k_i - k_j - ic} \prod_{j=1}^M \frac{k_i - \Lambda_j - ic/2}{k_i - \Lambda_j + ic/2} \quad i = 1, \dots, N \quad (2.15)$$

Here $\{\Lambda_j\}$ are a set of M unequal constants that arise from the solution of the eigenvalue equation (recall that M is the number of particles with spin- \downarrow). They are arbitrary in the solution of the first level equations, but will soon be restricted by the second level solution. If any two of $\{\Lambda_j\}$ are equivalent, then the solution becomes degenerate, as does the wavefunction. The meaning of these constants will be discussed below.

The second set of equations produced by solving the first level equations are solved in a similar manner, as given in Sutherland's paper. As our model only has a two-level spin state, the second level equations are our final equations. If we were using higher order spin states, we would have higher level equations again. Sutherland gives the second level Bethe ansatz equation to be

$$1 = \prod_{j=1}^N \frac{\Lambda_i - k_j - ic/2}{\Lambda_i - k_j + ic/2} \prod_{j \neq i}^M \frac{\Lambda_i - \Lambda_j + ic}{\Lambda_i - \Lambda_j - ic} \quad i = 1, \dots, M \quad (2.16)$$

The products may be extended to the entire range of variables, where the previously omitted terms in the products each contribute a negative sign. Our Bethe ansatz equations are thus

$$\begin{aligned} e^{ik_i L} &= - \prod_{j=1}^N \frac{k_i - k_j + ic}{k_i - k_j - ic} \prod_{j=1}^M \frac{k_i - \Lambda_j - ic/2}{k_i - \Lambda_j + ic/2} \quad i = 1, \dots, N \\ 1 &= - \prod_{j=1}^N \frac{\Lambda_i - k_j - ic/2}{\Lambda_i - k_j + ic/2} \prod_{j=1}^M \frac{\Lambda_i - \Lambda_j + ic}{\Lambda_i - \Lambda_j - ic} \quad i = 1, \dots, M \end{aligned} \quad (2.17)$$

Solutions of these equations give the energy and momentum of the system through equations 2.7.

2.2.4 Spin Rapidities

The variables $\{\Lambda_j\}$ are known as “spin rapidities”, and there exists one spin rapidity for each particle with spin- \downarrow . They have the same units as the pseudomomenta parameters $\{k_i\}$, but their physical meaning lies in the momentum description of spin-wave excitations. They do not directly contribute to the energy or the momentum of the system, although the energy associated with spin flipping affects the scattering process of the particles, impacting upon the distribution of the pseudomomentum parameters, as shown in the Bethe ansatz equations (equations 2.17). Spin rapidities may take complex values.

Here we introduce the term “charge rapidities” to denote the pseudomomenta. As pseudomomenta are not constrained to a particular particle, the set of charge rapidities as a whole describes the particles of the system, while the set of spin rapidities describes the spin-wave excitations in the system. We will often denote the two components of this system by charge and spin. In discussing fractional exclusion statistics, we even treat these components as individual pseudoparticles: each charge pseudoparticle has a particular pseudomomentum associated with it, while each spinon has a particular spin rapidity. It is important to remember that these are not physical particles.

2.2.5 The Ground State

Ignoring any external fields, the ground state of the gas has the lowest possible collection of pseudomomenta. This minimises the energy, which is the sum over all k_i^2 in the Dirac sea. From observing the effect a single spin rapidity has on the system, it can be seen that the ground state of the gas is a ferromagnetic state with $M = 0$, which is equivalent to the spinless case. A single particle with spin- \downarrow can lead to spin wave excitations, see [14, 15].

The Thermodynamic Bethe Ansatz

Towards the end of the 1960's, Yang and Yang [28] published a method for extracting thermodynamic information from the Bethe ansatz equations of the spinless boson. The method involved calculating particle and hole densities in the thermodynamic limit as $N \rightarrow \infty$ and $L \rightarrow \infty$, with $N/L = D$, a constant. From these densities, the Helmholtz free energy is calculated and minimised, resulting in an integral equation. The solution to this integral equation allows the calculation of the Helmholtz free energy, from which most relevant thermodynamic quantities can be calculated, at least in principle. This method is known as the thermodynamic Bethe ansatz (TBA).

Recently, Gu *et al.* [29] derived the TBA for the spinor Bose gas, using the string hypothesis to account for the spin rapidities. Here we follow their method closely, deriving the TBA utilising both the string hypothesis and a low temperature approximation.

3.1 The Spinor Boson

The thermodynamic Bethe ansatz may be derived for the spinor Bose gas by calculating the density of roots and holes for both pseudomomenta and spin rapidities. Unfortunately, whilst it has been shown that pseudomomenta values must be real, spin rapidities may be complex. Takahashi [19] has shown that spin rapidities in Bethe ansatz models conform to the so-called string hypothesis, where spin rapidities form bound states in 'strings' of varying lengths, with error $e^{-\epsilon N}$, where ϵ is a positive constant and N is the number of particles in the system. Thus, in the thermodynamic limit, we may assume this hypothesis. This allows us to calculate densities for strings of varying lengths, and thus obtain the density of spin rapidities.

Hatsugai *et al.* [30] have hypothesised that creating complex string rapidities requires more energy than real string rapidities, and thus that at low temperatures, only real spin rapidities occur. We begin by describing the string hypothesis, before proceeding to derive the thermodynamic Bethe ansatz assuming both the string hypothesis and real rapidities.

3.2 The String Hypothesis

The string hypothesis states that spin rapidities will join together in a 'bound state' to create 'strings'. Each of these strings contain a specific number of rapidities, which occur in complex conjugates about the same real coordinate. Each string may be written as $\lambda + (n + 1 - 2j)ic/2 + \mathcal{O}(\exp(-\epsilon N))$, where λ is the real component of the string, n is

$$\prod_{j=1}^n \frac{x + (n - 2j)ic/2}{x + (n + 2 - 2j)ic/2} = \frac{x - nic/2}{x + nic/2} \quad (3.4)$$

Applying these formulae to the first level BAE, and to the first product of the second level BAE, we arrive at

$$\begin{aligned} e^{ik_i L} &= - \prod_{j=1}^N \frac{k_i - k_j + ic}{k_i - k_j - ic} \prod_{na} \frac{k_i - \lambda_a^n - nic/2}{k_i - \lambda_a^n + nic/2} \quad i = 1, \dots, N \\ 1 &= - \prod_{\nu=1}^N \frac{\lambda_a^n - k_\nu - nic/2}{\lambda_a^n - k_\nu + nic/2} \prod_{mblj} \frac{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l + 2)ic/2}{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l - 2)ic/2} \\ &\quad \forall n, a \end{aligned} \quad (3.5)$$

The final step to simplifying these equations is to evaluate the products over j and l in the second level equation

$$\prod_{lj} \frac{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l + 2)ic/2}{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l - 2)ic/2} \quad \forall n, a \quad (3.6)$$

Multiplying over j first, we find that once again, denominators and numerators cancel, this time leaving the first two numerators and the final two denominators.

$$\begin{aligned} &\prod_{lj} \frac{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l + 2)ic/2}{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l - 2)ic/2} \\ &= \prod_l \frac{\lambda_a^n - \lambda_b^m + (2l + n - m)ic/2}{\lambda_a^n - \lambda_b^m + (2l - n - m)ic/2} \cdot \frac{\lambda_a^n - \lambda_b^m + (2l + n - m - 2)ic/2}{\lambda_a^n - \lambda_b^m + (2l - n - m - 2)ic/2} \end{aligned} \quad (3.7)$$

Next, we write this product out in full. Noticing that every numerator has a denominator which is its complex conjugate, we recombine the product by pairing these two components together.

$$\begin{aligned} &\prod_{lj} \frac{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l + 2)ic/2}{\lambda_a^n - \lambda_b^m + (n - 2j - m + 2l - 2)ic/2} \\ &= \prod_l \frac{\lambda_a^n - \lambda_b^m + (2l + n - m)ic/2}{\lambda_a^n - \lambda_b^m - (2l + n - m)ic/2} \cdot \frac{\lambda_a^n - \lambda_b^m + (2l + n - m - 2)ic/2}{\lambda_a^n - \lambda_b^m - (2l + n - m - 2)ic/2} \end{aligned} \quad (3.8)$$

This is as far as can be simplified for the moment. After taking logarithms, further simplifications may be made. Our final BAE using the string hypothesis are then

$$\begin{aligned} e^{ik_i L} &= - \prod_{j=1}^N \frac{k_i - k_j + ic}{k_i - k_j - ic} \prod_{na} \frac{k_i - \lambda_a^n - nic/2}{k_i - \lambda_a^n + nic/2} \quad i = 1, \dots, N \\ 1 &= - \prod_{\nu=1}^N \frac{\lambda_a^n - k_\nu - nic/2}{\lambda_a^n - k_\nu + nic/2} \\ &\quad \prod_{mbl} \frac{\lambda_a^n - \lambda_b^m + (2l + n - m)ic/2}{\lambda_a^n - \lambda_b^m - (2l + n - m)ic/2} \cdot \frac{\lambda_a^n - \lambda_b^m + (2l + n - m - 2)ic/2}{\lambda_a^n - \lambda_b^m - (2l + n - m - 2)ic/2} \\ &\quad \forall n, a \end{aligned} \quad (3.9)$$

3.4 Logarithmic Form

String Hypothesis

The BAE can be simplified by defining $S_a(x) = \frac{x+iac}{x-iac}$. Then we may write the equations as

$$\begin{aligned}
e^{ik_\nu L} &= - \prod_{j=1}^N S_1(k_\nu - k_j) \prod_{n,a} S_{-n/2}(k_\nu - \lambda_a^n) \quad \nu = 1, \dots, N \\
1 &= - \prod_{\nu=1}^N S_{-n/2}(\lambda_a^n - k_\nu) \\
&\quad \prod_{m,b,l} S_{(2l+n-m)/2}(\lambda_a^n - \lambda_b^m) S_{(2l+n-m-2)/2}(\lambda_a^n - \lambda_b^m) \quad \forall n, a \quad (3.10)
\end{aligned}$$

Start by taking the logarithm of both sides, allowing for a non-principal logarithm. Recall that $\ln(-1) = \pi i$. With I_ν and J_a^n integers, we obtain

$$\begin{aligned}
ik_\nu L &= (2\pi I_\nu + \pi)i + \sum_{j=1}^N \ln(S_1(k_\nu - k_j)) \\
&\quad + \sum_{n,a} \ln(S_{-n/2}(k_\nu - \lambda_a^n)) \quad \nu = 1, \dots, N \\
0 &= (2\pi J_a^n + \pi)i + \sum_{\nu=1}^N \ln(S_{-n/2}(\lambda_a^n - k_\nu)) + \sum_{m,b,l} (\ln(S_{(2l+n-m)/2}(\lambda_a^n - \lambda_b^m)) \\
&\quad + \ln(S_{(2l+n-m-2)/2}(\lambda_a^n - \lambda_b^m))) \quad \forall n, a \quad (3.11)
\end{aligned}$$

We now turn our attention to $\ln(S_a(x))$. By considering sine and cosine as the sum of exponentials, we can write \arctan as a logarithm.

$$\arctan(x) = \frac{1}{2i} \ln \left(\frac{1+ix}{1-ix} \right)$$

Using the identities $\arctan(x) = \pi/2 - \arctan(1/x)$ for $x > 0$ and $\arctan(x) = -\pi/2 - \arctan(1/x)$ for $x < 0$ we can rewrite this expression as

$$\ln \left(\frac{x+i}{x-i} \right) = i(\text{signum}(x)\pi - 2 \arctan(x))$$

Thus, we can write $\ln(S_a(x)) = i(\text{signum}(\frac{x}{ca})\pi - 2 \arctan(\frac{x}{ca}))$. Define $\Theta_a(x) = -2 \arctan(\frac{x}{ca})$. The Bethe ansatz equations then become

$$\begin{aligned}
k_\nu L &= (2\pi I_\nu + \pi) + \sum_{j=1}^N (\Theta_1(k_\nu - k_j) + \pi \text{signum}(k_\nu - k_j)) \\
&\quad + \sum_{n,a} (\Theta_{-n/2}(k_\nu - \lambda_a^n) - \pi \text{signum}(k_\nu - \lambda_a^n)) \quad \nu = 1, \dots, N
\end{aligned}$$

$$\begin{aligned}
 0 &= (2\pi J_a^n + \pi) + \sum_{\nu=1}^N (\Theta_{-n/2}(\lambda_a^n - k_\nu) - \pi \operatorname{signum}(\lambda_a^n - k_\nu)) \\
 &\quad + \sum_{m,b,l} ((\Theta_{(2l+n-m-2)/2}(\lambda_a^n - \lambda_b^m) + \pi \operatorname{signum}\left(\frac{\lambda_a^n - \lambda_b^m}{2l+n-m-2}\right)) \\
 &\quad + (\Theta_{(2l+n-m)/2}(\lambda_a^n - \lambda_b^m) + \pi \operatorname{signum}\left(\frac{\lambda_a^n - \lambda_b^m}{2l+n-m}\right))) \quad \forall n, a \quad (3.12)
 \end{aligned}$$

As I_ν and J_a^n are integers, adding some multiple of 2π to each of these equations only adjusts the integer, and does not change the actual form of the equations. Thus, we can drop the signums, effectively adding 2π for each negative signum term. The sums over constant π may then be performed. We shift J_a^n to the opposite side of the equation, which may also be seen as subtracting 2π sufficiently many times. Thus

$$\begin{aligned}
 k_\nu L &= 2\pi I_\nu + \pi + (N + M)\pi + \sum_{j=1}^N \Theta_1(k_\nu - k_j) + \sum_{n,a} \Theta_{-n/2}(k_\nu - \lambda_a^n) \quad \nu = 1, \dots, N \\
 2\pi J_a^n &= \pi + (N + M)\pi + \sum_{\nu=1}^N \Theta_{-n/2}(\lambda_a^n - k_\nu) \\
 &\quad + \sum_{m,b,l} (\Theta_{(2l+n-m)/2}(\lambda_a^n - \lambda_b^m) + \Theta_{(2l+n-m-2)/2}(\lambda_a^n - \lambda_b^m)) \quad \forall n, a \quad (3.13)
 \end{aligned}$$

Finally, we rename I_ν and J_a^n to incorporate the $(M + N + 1)\pi$ terms. We find that for $N + M$ odd, I_ν and J_a^n are integers, whereas for $N + M$ even, they are half-integral. Now

$$\begin{aligned}
 k_\nu L &= 2\pi I_\nu + \sum_{j=1}^N \Theta_1(k_\nu - k_j) + \sum_{n,a} \Theta_{-n/2}(k_\nu - \lambda_a^n) \quad \nu = 1, \dots, N \\
 2\pi J_a^n &= \sum_{\nu=1}^N \Theta_{-n/2}(\lambda_a^n - k_\nu) \\
 &\quad + \sum_{m,b,l} (\Theta_{(2l+n-m)/2}(\lambda_a^n - \lambda_b^m) + \Theta_{(2l+n-m-2)/2}(\lambda_a^n - \lambda_b^m)) \quad \forall n, a \quad (3.14)
 \end{aligned}$$

The last sum in the second level equation may be simplified by expanding the sum over l . Of the $m + 1$ terms present, the leading and trailing terms have coefficients of unity, whilst the inner terms have coefficients of 2. The leading term is of the form $\Theta_{(n+m)/2}$, whilst the trailing term is of the form $\Theta_{(n-m)/2}$. Next, noting that Θ_a is an odd function in a , terms of the form Θ_{-x} will cancel out terms of the form Θ_x , leaving a trailing term of the form $\Theta_{|n-m|/2}$ with coefficient unity. Note, however, that any terms of the form Θ_0 have a coefficient of zero, as Θ_0 is undefined and occurs only as a result of $\ln(S_0(x)) = \pi i$, which has already been absorbed into J_a^n . Thus, we rewrite the sum as a sum over all possible Θ_x , where the coefficient of each term is dependent upon x :

$$\begin{aligned}
 &\sum_{m,b,t \neq 0} A_{nmt} \Theta_{t/2}(\lambda_a^n - \lambda_b^m) \\
 A_{nmt} &= \begin{cases} 1 & \text{for } t = n + l, |n - l| \\ 2 & \text{for } t = n + l - 2, \dots, |n - l| + 2 \\ 0 & \text{otherwise} \end{cases} \quad (3.15)
 \end{aligned}$$

Our final version of the BAE then becomes

$$\begin{aligned}
k_\nu L &= 2\pi I_\nu + \sum_{j=1}^N \Theta_1(k_\nu - k_j) + \sum_{n,a} \Theta_{-n/2}(k_\nu - \lambda_a^n) \quad \nu = 1, \dots, N \\
2\pi J_a^n &= \sum_{\nu=1}^N \Theta_{-n/2}(\lambda_a^n - k_\nu) \\
&\quad + \sum_{m,b,t \neq 0} A_{nmt} \Theta_{t/2}(\lambda_a^n - \lambda_b^m) \quad \forall n, a
\end{aligned} \tag{3.16}$$

These are known as the Bethe ansatz equations in logarithmic form. In all, there are $M + N$ equations.

Low Temperature Approximation

To obtain the low temperature approximation from the string hypothesis equations, all that needs to be done is assume that only strings of length one exist, which means that all spin rapidities become real. The BAE become

$$\begin{aligned}
k_\nu L &= 2\pi I_\nu + \sum_{j=1}^N \Theta_1(k_\nu - k_j) + \sum_{\gamma=1}^M \Theta_{-1/2}(k_\nu - \lambda_\gamma) \quad \nu = 1, \dots, N \\
2\pi J_\gamma &= \sum_{\nu=1}^N \Theta_{-1/2}(\lambda_\gamma - k_\nu) + \sum_{\mu=1}^M \Theta_1(\lambda_\gamma - \lambda_\mu) \quad \gamma = 1, \dots, M
\end{aligned} \tag{3.17}$$

There are still $M + N$ equations in all.

3.5 Quantum Numbers

The numbers I_ν and J_a^n (or J_γ , for the low temperature approximation) can be thought of as quantum numbers for the model. By differencing two quantum numbers I_m and I_n ($m \neq n$), it can be seen that if $I_m = I_n$, then $k_m = k_n$, making the wavefunction degenerate. Thus, no two $\{I_\nu\}$ may be the same. Similarly for J_a^n and λ_a^n , and so our quantum numbers are well behaved (“good” quantum numbers).

The momentum for the system may be written in terms of these quantum numbers: $P = \sum_l k_l = (\sum_\nu I_\nu - \sum_{n,a} J_a^n) 2\pi/L$. It is easiest to work backwards rather than forwards to show this, remembering that $\Theta(x)$ is an odd function.

3.6 Integral Bethe Ansatz Equations

$\{I_\nu\}$ and $\{J_a^n\}$ play the role of quantum numbers for the charge and spin rapidities respectively (see Section 2.2.4 for a description of charge and spin). For a particular configuration, if an arbitrary quantum number is chosen, it is either occupied (in the set of quantum numbers for the system), or it is not. Following Gu *et al.* [29], we term the former a root, and the latter a hole. In the thermodynamic limit, the distribution of charge rapidities becomes dense, and so we introduce the density function of charge roots and holes. Define $\rho(k)$ and $\rho^h(k)$ to be the densities of charge roots and holes, respectively. For the spin rapidities, we define $\sigma_n(\lambda)$ and $\sigma_n^h(\lambda)$ to be the densities of the n -string (strings of length n) roots and holes in λ -space.

Treating I as a continuous function of k , for every integer that I increases, one more space for a root or a hole is created. Call the total number of particles and holes per unit length ϵ . Then we find

$$dI(k)/dk = L d\epsilon/dk = L(\rho(k) + \rho^h(k)). \quad (3.18)$$

Similarly, by treating J^n as a continuous function of λ , we have

$$(1/L) dJ^n(\lambda)/d\lambda = \sigma_n(k) + \sigma_n^h(k) \quad (3.19)$$

String Hypothesis

The next step is to take the continuous limit of the logarithmic form of the Bethe ansatz equations, changing to continuous I and J^n , and moving from summations to integrals. We treat all integrals as having limits of $-\infty$ to ∞ and suppress the limits in the equations, unless indicated otherwise.

$$\begin{aligned} \frac{I(k)}{L} &= \frac{k}{2\pi} - \frac{1}{2\pi} \int \Theta_1(k - k') \rho(k') dk' \\ &\quad - \sum_n \frac{1}{2\pi} \int \Theta_{-n/2}(k - \lambda') \sigma_n(\lambda') d\lambda' \end{aligned} \quad (3.20)$$

$$\begin{aligned} \frac{J^n(\lambda)}{L} &= \frac{1}{2\pi} \int \Theta_{-n/2}(\lambda - k') \rho(k') dk' \\ &\quad + \frac{1}{2\pi} \sum_{m,t \neq 0} A_{nmt} \int \Theta_{t/2}(\lambda - \lambda') \sigma_m(\lambda') d\lambda' \end{aligned} \quad (3.21)$$

We then differentiate both sides of the first level equation by k , and substitute for $dI(k)/dk$ from equation 3.18, whilst doing the same for the second level equations with λ and $dJ^n(\lambda)/d\lambda$. In this way

$$\begin{aligned} \rho(k) + \rho^h(k) &= \frac{1}{2\pi} + \int K_2(k - k') \rho(k') dk' \\ &\quad - \sum_n \int K_n(k - \lambda') \sigma_n(\lambda') d\lambda' \end{aligned} \quad (3.22)$$

$$\begin{aligned} \sigma_n(k) + \sigma_n^h(k) &= \int K_n(\lambda - k') \rho(k') dk' \\ &\quad - \sum_{m,t \neq 0} A_{nmt} \int K_t(\lambda - \lambda') \sigma_m(\lambda') d\lambda' \end{aligned} \quad (3.23)$$

where $K_n(x) = nc/2\pi(n^2c^2/4 + x^2)$. Note that all integrals are actually convolutions, and so the equations may be compactly written

$$\rho(k) + \rho^h(k) = \frac{1}{2\pi} + K_2(k) * \rho(k) - \sum_n K_n(k) * \sigma_n(k) \quad (3.24)$$

$$\sigma_n(k) + \sigma_n^h(k) = K_n(\lambda) * \rho(\lambda) - \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \sigma_m(\lambda) \quad (3.25)$$

with $*$ denoting integral convolution. These are known as the integral Bethe ansatz equations.

These densities are particularly useful, as all basic quantities can be written in terms

of them. For example, kinetic energy per unit length is given by $E_k/L = \int k^2 \rho(k) dk$, particle number per unit length by $D = N/L = \int \rho(k) dk$, and number of down spins by $M/L = \sum_n n \int \sigma_n(\lambda) d\lambda$.

Low Temperature Approximation

The integral BAE using the string hypothesis reduce simply to the low temperature approximation by once again assuming that only strings of length one exist. The above equations are modified thus:

$$\rho(k) + \rho^h(k) = \frac{1}{2\pi} + K_2(k) * \rho(k) - K_1(k) * \sigma(k) \quad (3.26)$$

$$\sigma(k) + \sigma^h(k) = K_1(\lambda) * \rho(\lambda) - K_2(\lambda) * \sigma(\lambda) \quad (3.27)$$

Equations for kinetic energy, particle density and the density of spin down particles are the same as above, reducing the sum for M/L to strings of length one only.

3.7 The Thermodynamic Bethe Ansatz

To obtain the conditions on the system for a state of thermodynamic equilibrium, we need to construct the Helmholtz free energy, and minimise it with regards to the density functions ρ and σ_n , constrained to equations 3.24 and 3.25. The Helmholtz free energy is used as it represents the free energy at constant temperature and volume (or in one dimension, length), which are the conditions under which we desire our equilibrium equations. We use the Helmholtz free energy from the grand partition function, $F = -k_B T \ln(\mathcal{Z})$. This gives the free energy F in terms of the energy E , the temperature T , the entropy S , and the number of particles N , where we use μ to denote the chemical potential. Since we are working in the thermodynamic limit, we deal with quantities per unit length, as the quantities themselves are meaningless (infinite). Once again, integration from $-\infty$ to ∞ is implied unless specified otherwise.

$$F/L = E/L - TS/L - \mu N/L \quad (3.28)$$

The String Hypothesis

The energy of the system is the sum of the kinetic energy and any energy arising from external fields, as indicated in the Hamiltonian for the system (equation 2.5). We thus use

$$\begin{aligned} E/L &= E_k/L + \Omega(2M/L - N/L) \\ &= \int k^2 \rho(k) dk + \Omega(2 \sum_n n \int \sigma_n(\lambda) d\lambda - \int \rho(k) dk) \\ &= \int (k^2 - \Omega) \rho(k) dk + \sum_n 2n\Omega \int \sigma_n(\lambda) d\lambda \end{aligned} \quad (3.29)$$

Here T is the temperature of the system and S/L the entropy per unit length. μ is the chemical potential, which does not depend on the density functions. We use the equation for N/L given above (Section 3.6).

In their seminal paper, Yang and Yang [28] developed an expression for the entropy of the spinless Bose gas at finite temperature by constructing a statistical weight, considering

all possible arrangements of charge particles. Following the same method and treating spin rapidities as extra particles (as discussed in Section 2.2.4), the entropy of the spinor boson system is given by

$$\begin{aligned} S/L &= \int [(\rho + \rho^h) \ln(\rho + \rho^h) - \rho \ln(\rho) - \rho^h \ln(\rho^h)] dk \\ &\quad + \sum_n \int [(\sigma_n + \sigma_n^h) \ln(\sigma_n + \sigma_n^h) - \sigma_n \ln(\sigma_n) - \sigma_n^h \ln(\sigma_n^h)] d\lambda \end{aligned} \quad (3.30)$$

The first integral is the entropy of the spinless boson gas as presented by Yang and Yang. It is logical that the addition of extra quantum numbers will add to the entropy in a similar manner.

When we minimise this quantity with respect to ρ and σ_n , it is useful to have these terms cast in a different way through simple logarithm manipulations.

$$\begin{aligned} &(\rho + \rho^h) \ln(\rho + \rho^h) - \rho \ln(\rho) - \rho^h \ln(\rho^h) \\ &\quad \equiv (\rho + \rho^h) \ln(1 + \rho/\rho^h) + \rho \ln(\rho^h/\rho) \\ &(\sigma_n + \sigma_n^h) \ln(\sigma_n + \sigma_n^h) - \sigma_n \ln(\sigma_n) - \sigma_n^h \ln(\sigma_n^h) \\ &\quad \equiv (\sigma_n + \sigma_n^h) \ln(1 + \sigma_n/\sigma_n^h) + \sigma_n \ln(\sigma_n^h/\sigma_n) \end{aligned} \quad (3.31)$$

Before proceeding, let us define some new quantities:

$$\begin{aligned} \kappa(k) &= e^{\epsilon(k)/T} = \rho^h(k)/\rho(k) \\ \eta_n(\lambda) &= e^{\xi_n(\lambda)/T} = \sigma_n^h(\lambda)/\sigma_n(\lambda) \end{aligned} \quad (3.32)$$

We are now ready to minimise the free energy, which may be written in full as

$$\begin{aligned} F/L &= \int (k^2 - \Omega) \rho(k) dk + \sum_n 2n\Omega \int \sigma_n(\lambda) d\lambda \\ &\quad - T \int [(\rho + \rho^h) \ln(1 + \kappa^{-1}) + \rho \ln(\kappa)] dk \\ &\quad - T \sum_n \int [(\sigma_n + \sigma_n^h) \ln(1 + \eta_n^{-1}) + \sigma_n \ln(\eta_n)] d\lambda - \mu \int \rho(k) dk \\ &= \int [(k^2 - \Omega - \mu - T \ln(\kappa)) \rho(k) - T(\rho + \rho^h) \ln(1 + \kappa^{-1})] dk \\ &\quad + \sum_n \int [(2n\Omega - T \ln(\eta_n)) \sigma_n(\lambda) - T(\sigma_n + \sigma_n^h) \ln(1 + \eta_n^{-1})] d\lambda \end{aligned} \quad (3.33)$$

We use equations 3.24 and 3.25 as constraints on ρ and σ_n . We now use the variational method, varying F by ρ and σ_m , such that $\delta F/L = A\delta\rho + \sum_n B_n\delta\sigma_n = 0$. A and B_n represent expressions multiplying the variances in ρ and σ_n respectively, and by the independence of ρ and σ_m , A and B_m must be zero.

$$\begin{aligned} \delta F/L &= \int [(k^2 - \Omega - \mu - T \ln(\kappa)) \delta\rho(k) - T\delta[(\rho + \rho^h) \ln(1 + \kappa^{-1})]] dk \\ &\quad + \sum_n \int [(2n\Omega - T \ln(\eta_n)) \delta\sigma_n(\lambda) - T\delta[(\sigma_n + \sigma_n^h) \ln(1 + \eta_n^{-1})]] d\lambda \end{aligned} \quad (3.34)$$

Because K_n is an even function, from manipulating the order of integration and renaming dummy variables, it can be shown that $\delta[(K_n(k) * \rho(k)) \ln(1 + \eta_n^{-1})] = K_n(k) * [\ln(1 + \eta_n^{-1})] \delta(\rho(k))$. We thus have

$$\begin{aligned} \delta[(\rho + \rho^h) \ln(1 + \kappa^{-1})] &= K_2(k) * \ln(1 + \kappa^{-1}) \delta\rho(k) - \sum_n K_n(k) * \ln(1 + \kappa^{-1}) \delta\sigma_n(k) \\ \delta[(\sigma_n + \sigma_n^h) \ln(1 + \eta_n^{-1})] &= K_n(\lambda) * \ln(1 + \eta_n^{-1}) \delta\rho(\lambda) \\ &\quad - \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \ln(1 + \eta_n^{-1}) \delta\sigma_m(\lambda) \end{aligned} \quad (3.35)$$

Combining these, we obtain

$$\begin{aligned} \delta F/L &= \int [(k^2 - \Omega - \mu - T \ln(\kappa)) \delta\rho(k) - T K_2(k) * \ln(1 + \kappa^{-1}) \delta\rho(k) \\ &\quad + T \sum_n K_n(k) * \ln(1 + \kappa^{-1}) \delta\sigma_n(k)] dk \\ &\quad + \sum_n \int [(2n\Omega - T \ln(\eta_n)) \delta\sigma_n(\lambda) - T K_n(\lambda) * \ln(1 + \eta_n^{-1}) \delta\rho(\lambda) \\ &\quad + T \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \ln(1 + \eta_n^{-1}) \delta\sigma_m(\lambda)] d\lambda \end{aligned} \quad (3.36)$$

Changing the dummy variables of summation for the final term, swapping m and n , then rearranging and changing dummy variables of integration, we arrive at

$$\begin{aligned} \delta F/L &= \int \left[k^2 - \Omega - \mu - T \ln(\kappa) - T K_2(k) * \ln(1 + \kappa^{-1}) \right. \\ &\quad \left. - T \sum_n K_n(k) * \ln(1 + \eta_n^{-1}) \right] \delta\rho(k) dk \\ &\quad + \sum_n T \int \left[\frac{2n\Omega}{T} - \ln(\eta_n) + K_n(\lambda) * \ln(1 + \kappa^{-1}) \right. \\ &\quad \left. + \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \ln(1 + \eta_m^{-1}) \right] \delta\sigma_n(\lambda) d\lambda \end{aligned} \quad (3.37)$$

By the independence of $\delta\rho(k)$ and $\delta\sigma_n(\lambda)$, we thus obtain the following two conditions for $\delta F/L = 0$.

$$\begin{aligned} k^2 - \Omega - \mu - T \ln(\kappa) - T K_2(k) * \ln(1 + \kappa^{-1}) - T \sum_n K_n(k) * \ln(1 + \eta_n^{-1}) &= 0 \\ \frac{2n\Omega}{T} - \ln(\eta_n) + K_n(\lambda) * \ln(1 + \kappa^{-1}) + \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \ln(1 + \eta_m^{-1}) &= 0 \end{aligned} \quad (3.38)$$

These become our TBA equations.

$$T \ln(\kappa) = \epsilon(k) = k^2 - \Omega - \mu - T K_2(k) * \ln(1 + \kappa^{-1}) - T \sum_n K_n(k) * \ln(1 + \eta_n^{-1}) \quad (3.39)$$

$$\ln(\eta_n) = \frac{\xi_n(\lambda)}{T} = \frac{2n\Omega}{T} + K_n(\lambda) * \ln(1 + \kappa^{-1}) + \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \ln(1 + \eta_m^{-1}) \quad (3.40)$$

Note that this second equation actually represents an infinite number of equations, one for each string length n . This provides difficulties for attempting numerical simulations, as the number of variables to solve the coupled integral equations over is prohibitive, and

thus must be terminated at some point.

Low Temperature Approximation

Once again, the low temperature approximation version of the TBA equations can be obtained by assuming strings of length one only. The TBA equations in a low temperature approximation are:

$$T \ln(\kappa) = \epsilon(k) = k^2 - \Omega - \mu - TK_2(k) * \ln(1 + \kappa^{-1}) - TK_1(k) * \ln(1 + \eta^{-1}) \quad (3.41)$$

$$\ln(\eta) = \frac{\xi(\lambda)}{T} = \frac{2\Omega}{T} + K_1(\lambda) * \ln(1 + \kappa^{-1}) + K_2(\lambda) * \ln(1 + \eta^{-1}) \quad (3.42)$$

3.8 Alternate Form of the Thermodynamic Bethe Ansatz Equations

The second TBA equation using the string hypothesis (equation 3.40) may be recast in another fashion which eliminates the infinite sums. Start by writing $\ln(\eta_n) = \ln(1 + \eta_n) - \ln(1 + \eta_n^{-1})$, and define $K_0(x) = \delta(x)$. Then

$$\ln(1 + \eta_n) = K_n(\lambda) * \ln(1 + \kappa^{-1}) + 2n\Omega/T + \sum_{m,t} A_{nmt} K_t(\lambda) * \ln(1 + \eta_m^{-1}) \quad (3.43)$$

Using the distributive property of convolutions, we can expand the sum over t , and change notation slightly to simplify the expression visually. This also has the effect of simplifying the Fourier transforms which we will take later. We write $\sum_t A_{nmt} K_t(\lambda) = K_{n+m} + 2K_{n+m-2} + \dots + 2K_{|n-m|+2} + K_{|n-m|} = A_{nm}$. So

$$\ln(1 + \eta_n) = K_n(\lambda) * \ln(1 + \kappa^{-1}) + 2n\Omega/T + \sum_m A_{nm} * \ln(1 + \eta_m^{-1}) \quad (3.44)$$

By taking Fourier transforms, it is possible to calculate an inverse for A_{nm} in w -space, which obeys the relations

$$\begin{aligned} \sum_{m=1}^{\infty} \hat{A}_{nm}^{-1}(w) \hat{A}_{mn'}(w) &= \delta_{nn'} \\ \sum_{m=1}^{\infty} A_{nm}^{-1}(\lambda) * A_{mn'}(\lambda) &= \delta_{nn'} \delta(\lambda) \\ \sum_{m=1}^{\infty} A_{nm}^{-1}(\lambda) * K_n(\lambda) &= \delta_{n1} \frac{1}{2c \cosh(\pi\lambda/c)} \\ \sum_{m=1}^{\infty} A_{nm}^{-1}(\lambda) * m &= 0 \end{aligned}$$

We find that $A_{nm}^{-1}(\lambda) = \delta(\lambda) \delta_{nm} - \frac{1}{2c \cosh(\pi\lambda/c)} (\delta_{n,m+1} + \delta_{n,m-1})$.

Using these relations, we convolute equation 3.44 with $A_{nl}^{-1}(\lambda)$, and sum over $n = 1 \dots \infty$. By exploiting the fact that the convolution operator is associative, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} A_{nl}^{-1} * \ln(1 + \eta_n) &= \sum_{n=1}^{\infty} A_{nl}^{-1} * K_n(\lambda) * \ln(1 + \kappa^{-1}) + \sum_{n=1}^{\infty} A_{nl}^{-1} * 2n\Omega/T \\
&\quad + \sum_m \sum_{n=1}^{\infty} A_{nl}^{-1} * A_{nm} * \ln(1 + \eta_m^{-1}) \\
&= \delta_{l1} \frac{1}{2c \cosh(\pi\lambda/c)} * \ln(1 + \kappa^{-1}) + \sum_m \delta_{lm} \delta(\lambda) * \ln(1 + \eta_m^{-1}) \\
&= \delta_{l1} \frac{1}{2c \cosh(\pi\lambda/c)} * \ln(1 + \kappa^{-1}) + \ln(1 + \eta_l^{-1}) \tag{3.45}
\end{aligned}$$

We now use the definition of A_{mn}^{-1} to arrive at

$$\begin{aligned}
\ln(\eta_l) - \frac{1}{2c \cosh(\pi\lambda/c)} * \ln(1 + \eta_{l+1}) - (1 - \delta_{l1}) \frac{1}{2c \cosh(\pi\lambda/c)} * \ln(1 + \eta_{l-1}) \\
= \delta_{l1} \frac{1}{2c \cosh(\pi\lambda/c)} * \ln(1 + \kappa^{-1}) \tag{3.46}
\end{aligned}$$

This separates into two equations:

$$\begin{aligned}
\ln(\eta_1) &= \frac{1}{2c \cosh(\pi\lambda/c)} * \ln[(1 + \kappa^{-1})(1 + \eta_2)] \\
\ln(\eta_n) &= \frac{1}{2c \cosh(\pi\lambda/c)} * \ln[(1 + \eta_{n+1})(1 + \eta_{n-1})] \quad \text{for } n > 1 \tag{3.47}
\end{aligned}$$

Note that there is no low temperature approximation equivalent of this form, which explicitly relies upon the existence of an infinite number of possible string lengths. These equations are concluded by the following asymptotic condition

$$\lim_{n \rightarrow \infty} \ln(\eta_n)/n = \frac{2\Omega}{T} \tag{3.48}$$

which has been derived by Takahashi [31].

Finally, we present the thermodynamic Bethe ansatz equations in full

$$\begin{aligned}
T \ln(\kappa) &= \epsilon(k) = k^2 - \Omega - \mu - TK_2(k) * \ln(1 + \kappa^{-1}) - T \sum_n K_n(k) * \ln(1 + \eta_n^{-1}) \\
\ln(\eta_1) &= \frac{1}{2c \cosh(\pi\lambda/c)} * \ln[(1 + \kappa^{-1})(1 + \eta_2)] \\
\ln(\eta_n) &= \frac{1}{2c \cosh(\pi\lambda/c)} * \ln[(1 + \eta_{n+1})(1 + \eta_{n-1})] \quad \text{for } n > 1
\end{aligned}$$

complete with asymptotic condition

$$\lim_{n \rightarrow \infty} \ln(\eta_n)/n = \frac{2\Omega}{T} \tag{3.49}$$

3.9 The Helmholtz Free Energy

If the solution $\epsilon(k)$ can be found to the TBA equations, then the Helmholtz free energy may be calculated. As we now have a fixed number of particles, instead of calculating the

Helmholtz free energy from the grand partition function \mathcal{Z} , we use the partition function Z , obtaining: $F = -k_B T \ln(Z) = E - TS$. Although we do not change the notation to account for this, all references to the (Helmholtz) free energy from now on will use this definition. We present here a derivation of the relation.

Firstly

$$\begin{aligned}
 F/L &= E/L - TS/L \\
 &= \int (k^2 - \Omega) \rho(k) dk + \sum_n 2n\Omega \int \sigma_n d\lambda \\
 &\quad - T \int [(\rho + \rho^h) \ln(\rho + \rho^h) - \rho \ln(\rho) - \rho^h \ln(\rho^h)] dk \\
 &\quad - T \sum_n \int [(\sigma_n + \sigma_n^h) \ln(\sigma_n + \sigma_n^h) - \sigma_n \ln(\sigma_n) - \sigma_n^h \ln(\sigma_n^h)] d\lambda \\
 &= \int (k^2 - \Omega) \rho(k) dk + \sum_n 2n\Omega \int \sigma_n d\lambda \\
 &\quad - T \int [(\rho + \rho^h) \ln(1 + \kappa^{-1}) + \rho \ln(\kappa)] dk \\
 &\quad - T \sum_n \int [(\sigma_n + \sigma_n^h) \ln(1 + \eta_n^{-1}) + \sigma_n \ln(\eta_n)] d\lambda \tag{3.50}
 \end{aligned}$$

Substitute equations 3.24 and 3.25 in here to give

$$\begin{aligned}
 F/L &= \int (k^2 - \Omega - T \ln(\kappa)) \rho(k) dk + \sum_n \int (2n\Omega - T \ln(\eta_n)) \sigma_n d\lambda \\
 &\quad - T \int \left(\frac{1}{2\pi} + K_2(k) * \rho(k) - \sum_n K_n(k) * \sigma_n(k) \right) \ln(1 + \kappa^{-1}) dk \\
 &\quad - T \sum_n \int \left(K_n(\lambda) * \rho(\lambda) - \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \sigma_m(\lambda) \right) \ln(1 + \eta_n^{-1}) d\lambda \tag{3.51}
 \end{aligned}$$

We now use the first TBA equation (equation 3.39) to substitute for $k^2 - \Omega - T \ln(\kappa)$, and reverse the order of integration on various convolutions in order to cancel terms, with

$$\begin{aligned}
 F/L &= \int (\mu + TK_2(k) * \ln(1 + \kappa^{-1}) + T \sum_n K_n(k) * \ln(1 + \eta_n^{-1})) \rho(k) dk \\
 &\quad + \sum_n \int (2n\Omega - T \ln(\eta_n)) \sigma_n d\lambda \\
 &\quad - T \int \left(\frac{1}{2\pi} + K_2(k) * \rho(k) - \sum_n K_n(k) * \sigma_n(k) \right) \ln(1 + \kappa^{-1}) dk \\
 &\quad - T \sum_n \int \left(K_n(\lambda) * \rho(\lambda) - \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \sigma_m(\lambda) \right) \ln(1 + \eta_n^{-1}) d\lambda \\
 &= \int \mu \rho(k) dk + \sum_n \int (2n\Omega - T \ln(\eta_n)) \sigma_n d\lambda \\
 &\quad - T \int \left(\frac{1}{2\pi} - \sum_n K_n(k) * \sigma_n(k) \right) \ln(1 + \kappa^{-1}) dk \\
 &\quad + T \sum_n \int \left(\sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \sigma_m(\lambda) \right) \ln(1 + \eta_n^{-1}) d\lambda \tag{3.52}
 \end{aligned}$$

Next, substitute for the second TBA equation (equation 3.40) to give

$$\begin{aligned}
F/L &= \mu \frac{N}{L} + \sum_n \int \left(2n\Omega - TK_n(\lambda) * \ln(1 + \kappa^{-1}) - 2n\Omega \right) \sigma_n \\
&\quad - \sum_n \int \left(T \sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \ln(1 + \eta_m^{-1}) \right) \sigma_n d\lambda \\
&\quad - T \int \left(\frac{1}{2\pi} - \sum_n K_n(k) * \sigma_n(k) \right) \ln(1 + \kappa^{-1}) dk \\
&\quad + T \sum_n \int \left(\sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \sigma_m(\lambda) \right) \ln(1 + \eta_n^{-1}) d\lambda \\
&= \mu \frac{N}{L} - T \sum_n \int \left(\sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \ln(1 + \eta_m^{-1}) \right) \sigma_n d\lambda - \frac{T}{2\pi} \int \ln(1 + \kappa^{-1}) dk \\
&\quad + T \sum_n \int \left(\sum_{m,t \neq 0} A_{nmt} K_t(\lambda) * \sigma_m(\lambda) \right) \ln(1 + \eta_n^{-1}) d\lambda \\
&= \mu \frac{N}{L} - \frac{T}{2\pi} \int \ln(1 + \kappa^{-1}) dk \tag{3.53}
\end{aligned}$$

Thus we can write

$$F = \mu N - \frac{TL}{2\pi} \int_{-\infty}^{\infty} \ln(1 + e^{-\epsilon(k)/T}) dk \tag{3.54}$$

The same result holds in the low temperature approximation.

We note that $F = \mu N - PV$ also holds (where V is volume), and so we may write the pressure of the system as

$$P = \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln(1 + e^{-\epsilon(k)/T}) dk \tag{3.55}$$

The quantity of most interest that comes from the free energy is the heat capacity at constant temperature, given by

$$S = - \left[\frac{\partial F}{\partial T} \right]_{V,N} \quad C_v = T \left[\frac{\partial S}{\partial T} \right]_{V,N} \tag{3.56}$$

where S is the entropy of the system.

3.10 Summary

Let us briefly pause to recapitulate on what we have covered so far.

Last chapter, we started with the Hamiltonian for the spinor Bose gas, and used the Bethe ansatz (trial wavefunction) to find a solution. From this ansatz, we obtained an expression for energy and momentum in terms of the pseudomomenta variables k_i .

We then made sure that the coefficients of each term in the ansatz obeyed certain criteria, namely symmetry under particle interchange, periodic boundary conditions, continuity, and discontinuity in the derivative at the δ -functions. These four criteria led to a series of conditions being placed on the pseudomomenta, called the Bethe ansatz equations (equations 2.17).

In this chapter, we simultaneously developed two forms of the Bethe ansatz equations,

one in a low temperature approximation with real string rapidities, and one assuming the string hypothesis. We started by taking the logarithm of the Bethe ansatz equations, taking into account the possibility of non-principal values, to arrive in their logarithmic form (equations 3.17 and 3.16). Next, we took the thermodynamic limit, and found the Bethe ansatz equations in integral form (equations 3.24 and 3.24, equations 3.27), which describe the root density of the charge and spin pseudoparticles.

Next, we constructed the Helmholtz free energy for the system, which we then proceeded to minimise with regards to the charge and spin root densities. By minimising the free energy, we obtained conditions on the system when it is in thermal equilibrium. We obtained the thermodynamic Bethe ansatz equations, both using the string hypothesis (equations 3.39 and 3.40), and in the low temperature approximation (equations 3.41 and 3.42). We then presented an alternate form of the TBA equations using the string hypothesis (equations 3.49). Finally, we derived an expression for the Helmholtz free energy in terms of the solution to the TBA equations (equation 3.54).

The significance of these equations is that if the thermodynamic Bethe ansatz equations can be solved along with the integral form of the Bethe ansatz equations, then the system is completely described for thermal equilibrium, and the excited states of the system may be characterised. Furthermore, once these equations are solved, all of the thermodynamics of the system may then be calculated.

In the next chapter, we take a look at fractional exclusion statistics, before returning to analyse the BAE and TBA equations in some limiting cases in Chapter 5.

Fractional Exclusion Statistics and the Thermodynamic Bethe Ansatz

A statistical approach is often adopted in thermodynamics, rather than attempts to follow each and every particle. Because of the number of particles usually involved, using probabilities and statistics is known to be a relatively precise way of treating systems. The most basic statistical model is that of Maxwell-Boltzmann statistics, which assumes that all particles are distinguishable. Starting with the Gibbs paradox, and following the advent of quantum mechanics, it was decided that identical particles were indistinguishable, and so quantum statistics was born. There are two main components to quantum statistical mechanics: Fermi-Dirac statistics, which assumes that in any state, there may only be one particle at a time, and Bose-Einstein statistics, which assumes that there may be any number of particles in the same state.

In between these two extremes lies an area known as fractional exclusion statistics. Here, the number of particles that may occupy a state is not necessarily infinity nor one, and may behave unexpectedly. “Fractional Statistics and Quantum Theory” by Khare [32] presents an overview of different types of fractional statistics, including Haldane exclusion statistics, Gentile statistics, and Polychronakos statistics in Chapter 5. The book as a whole deals with anyonic fractional statistics.

Another way of thinking about the statistics of particles is to say that it defines the phase change that a wavefunction gains upon the interchange of two particles. For bosons, this phase change is zero, whilst for fermions, this phase change is -1 . When dealing with anyons however, the phase change upon particle interchange may be an arbitrary $e^{i\theta}$, $0 \leq \theta \leq \pi$. As anyons exist in two dimensions, whether to take the positive of this phase change or the negative depends on whether the interchange occurs clockwise or counterclockwise.

Specifically with regards to the Bethe ansatz model, a branch of fractional exclusion statistics known as Haldane exclusion statistics has been shown to apply to the spinless boson. In this chapter, we present Haldane exclusion statistics, some thermodynamics which may be derived from the principle, and discuss its implementation with regards to the spinless boson. We then discuss the spinor boson with regards to Haldane exclusion statistics.

4.1 Haldane Exclusion Statistics

In 1991, Haldane [22] introduced a generalisation of the Pauli exclusion principle. The basic tenet of this generalisation is the counting of the dimensionality of Hilbert spaces.

Here we present the construction of this generalisation as discussed in Haldane's original paper.

Consider a Hilbert space \mathcal{H}_a of states of a single particle of species a in a quantum system, and assume that $\dim[\mathcal{H}_a] = d_a$ is finite and extensive. The Hilbert space is then spanned by d_a basis functions, $\phi_\nu^a(r)$, $\nu = 1 \dots d_a$. Note that identical particles are not necessarily in the same species.

Now consider the wave function of an N -particle system. Denote the coordinates and species of each of these particles by $\{r_i, a_i; i = 1, \dots, N\}$, and call N_a the number of particles within species a . If the coordinates of the $N - 1$ particles with labels $j \neq i$ are held fixed, then the wavefunction Ψ may be expanded in a basis of the wavefunctions of particle i , such that

$$\Psi = \sum_{\mu} A_{\mu}(\{r_j, a_j; j \neq i\}) \phi_{\mu}^{a_i}(r_i)$$

The set of functions $\phi_{\nu}^{a_i}$ span a one-particle Hilbert space \mathcal{H}_{a_i} , which has dimensions $d_{a_i}(\{N_{a_i}\})$. This dimension must be independent of the coordinates of the particles labelled $j \neq i$, and must be the same for all identical particles of the same species, depending only on the boundary conditions of the system, and the distribution of other particles within the system $\{N_{a_i}\}$.

Keeping the boundary conditions and volume of the system constant, d_{a_i} will change as particles are added or removed from the system. This provides the basic notion of statistics developed by Haldane. Assuming the relation between the change in dimension and the change in particle number is linear, we define the *statistical interaction* g_{ab} to account for these changes through the difference relation

$$\Delta d_a = - \sum_b g_{ab} \Delta N_b \quad (4.1)$$

where $\{\Delta N_b\}$ is a set of allowed changes to the particle numbers of each species, and the sum runs over all particle species. This essentially implies that as a particle of species a is added to the system, the dimensionality of the Hilbert space of species b may change, according to this relation. This equation is essentially the definition of Haldane exclusion statistics (sometimes called Haldane-Wu exclusion statistics). In differential form, this equation may be written

$$\frac{\partial d_a}{\partial N_b} = -g_{ab} \quad (4.2)$$

This statistical interaction has a number of properties. To start with, g_{ab} is not necessarily symmetrical. In order for a thermodynamic limit to exist using extensive Hilbert space dimensions, g_{ab} must be independent of the number of particles of each species within the system. Curiously, the existence of a thermodynamic limit also requires that g_{ab} is rational, so that the limit may be achieved through a sequence of finite increases in size and particle numbers.

This definition has two limiting cases which correspond to Bose-Einstein statistics and Fermi-Dirac statistics. Letting $g_{ab} = 0$ implies that the dimensionality of the Hilbert spaces doesn't change as particles are added, essentially allowing an infinite number of particles of each species in the system. This corresponds to Bose-Einstein statistics. On the other hand, Fermi-Dirac statistics are achieved by $g_{ab} = \delta_{ab}$, and so adding a particle of species a decreases the number of particles of species a that may be added to the system by one. It also means that species b is unaffected by the addition of particles of species a .

A few theoretical models have been shown to obey Haldane exclusion statistics, including Laughlin quasiparticles which have $g = 1/m$, and 1D spinons, which have $g = 1/2$, where only one species exists in each of these models [33]. More importantly, it has also been shown that the topological excitations of the fractional quantum Hall effect can be explained by Haldane exclusion statistics, giving rise to the same results derived from the braid group, which have been experimentally verified.

This way of defining statistics can lead to some strange consequences, the most obvious of which is fractional dimensions. There are some more esoteric consequences that arise from the definition, such as negative probabilities, which have even stranger resolutions. Khare [32] gives a good overview of Haldane Exclusion Statistics in Chapter 5 of his book.

Particles which obey Haldane exclusion statistics have been coined “exclusons”. When the statistics are such that there are interspecies interactions, the particles are said to have “mutual statistics”, while if particles only interact with other particles of the same species, they are said to have “non-mutual statistics”, and are called “g-ons”.

4.2 Thermodynamics from Haldane's Principle

In this section, we take Haldane exclusion statistics, and combine it with conventional thermodynamics to find how the statistical interaction affects thermodynamic properties. Rather than giving derivations of the following, we refer to the papers in which they were published.

In calculating the number of different possible states of N particles given a group of G states in quantum statistical mechanics, thermodynamic textbooks give the statistical weight as

$$W_b = \frac{(G + N - 1)!}{N!(G - 1)!} \quad \text{for bosons}$$

$$W_f = \frac{G!}{N!(G - N)!} \quad \text{for fermions}$$

Through a simple interpolation, Wu [34] has shown that fractional statistics for one species may be written as

$$W = \frac{(G + (N - 1)(1 - \alpha))!}{N!(G - \alpha N - (1 - \alpha))!}$$

Note that Wu has used α to represent the statistical interaction. Wu also gives the result

$$W = \prod_i \frac{(G_i + N_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij}))}{(N_i)!(G_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij}))!}$$

for multiple species.

From the statistical weight, Wu derived the grand partition function, using the ideal gas assumption that the energy is a simple sum over particle numbers times energy eigenvalues. In the case of non-mutual statistics (statistics only between particles of the same species), where $\alpha_{ij} = \alpha\delta_{ij}$ and $\mu_i = \mu$, a statistical distribution may be derived of the form

$$n_i = \frac{1}{w(e^{(\epsilon_i - \mu)/k_B T}) + \alpha} \quad (4.3)$$

where $w(x)$ satisfies $w(x)^\alpha(1+w(x))^{1-\alpha} = x$. ϵ_i refers to the energy of species i . Note that $\alpha = 0, 1$ reclaims the well known Bose-Einstein and Fermi-Dirac statistics, respectively. Wu further went on to show how a Fermi-step like structure occurs at $T = 0$, with distributions $n_i = 1/\alpha$ for $\epsilon_i < E_F$, and $n_i = 0$ for $\epsilon_i > E_F$.

Nayak and Wilczek [33] built upon the work of Wu in their paper, deriving the same relations and also finding the Fermi surface. They coined the name “g-ons” to describe particles that obey non-mutual Haldane exclusion statistics, after the symbol often used to denote the statistical interaction. In their paper, they went on to derive an expansion of energy and chemical potential in powers of temperature using a Sommerfeld Lemma style approach (see Appendix A), and discussed the duality property of g-ons, whereby particles obeying statistical interaction $g = 1/\beta$ are strongly related to particles obeying statistical interaction $g = \beta$. Khare [32] also discusses this property in Chapter 5 of his book.

Later, Isakov *et al.* [23] furthered the work of Nayak and Wilczek. They used Sommerfeld expansions to calculate expansions for the chemical potential and specific heat at low temperatures, as well as calculating cluster and virial expansions, all assuming a general dispersion relation $\epsilon(p) = ap^\sigma$. The chemical potential is

$$\mu = \mu_0 \left[1 - \frac{\pi^2}{6} g \left(\frac{D}{\sigma} - 1 \right) \left(\frac{k_B T}{\mu_0} \right)^2 + \dots \right] \quad (4.4)$$

where $\mu_0 = (g\rho/\Delta)^{\sigma/D}$ is the chemical potential at $T = 0$, and $\rho = N/V$ is the particle density. D is the dimension of the system, and $\Delta = a^{-D/\sigma} / [(2\sqrt{\pi}\hbar)^D \Gamma(1 + D/2)]$. The specific heat is

$$\frac{C_{V,N}}{V} = \frac{gD\rho}{\sigma} \left(\frac{k_B T}{\mu_0} \right) \left[\frac{\pi^2}{3} + 6\zeta(3)(1-g) \left(\frac{D}{\sigma} - 1 \right) \frac{k_B T}{\mu_0} + \dots \right] \quad (4.5)$$

where $\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$ is the Riemann zeta function.

4.3 FES of the Spinless Boson

In 1994, Wu and Bernard showed that the spinless boson in the thermodynamic limit obeys Haldane exclusion statistics, with different momenta giving rise to different species. Here we reproduce their derivation.

Recall the integral Bethe ansatz equations from Chapter 3. When $M = 0$, there are no spin down particles, and so $M/L = \sum_n n \int \sigma_n(\lambda) d\lambda = 0$. Since $\sigma_n \geq 0$, for the sum of the integrals to equal zero, $\sigma_n = 0 \forall n$. Then the first level Bethe ansatz equation, which corresponds to the spinless boson model, is

$$\begin{aligned} \rho(k) + \rho^h(k) &= \frac{1}{2\pi} + K_2(k) * \rho(k) \\ &= \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2c}{c^2 + (k - k')^2} \rho(k') dk' \end{aligned} \quad (4.6)$$

Recalling that $(\rho + \rho_h)(k)$ actually refers to the number of particles and holes per unit length between k and $k + dk$, we now derive this same quantity from a state counting picture. Call the total number of allowed particle states of species i , $D_i(\{N_j\})$. Then this

must be equal to the number of particles of species i already present, plus the dimensionality of the Hilbert space of these particles, minus one (as dimensionality one means that there is no room for more particles). In an equation, $D_i(\{N_j\}) = N_i + d_i(\{N_j\}) - 1$.

Now, integrate equation 4.2 to calculate $d_i(\{N_j\})$.

$$d_i(\{N_j\}) = G_i^0 - \sum_j g_{ij} N_j$$

$G_i^0 \equiv d_i(\{0\})$ is a constant, which is interpreted as one more than the number of available single particles states for species i when there are no particles in the system. Combining these, we have

$$D_i(\{N_j\}) = N_i + G_i^0 - \sum_j g_{ij} N_j - 1 \quad (4.7)$$

Dividing by L , the length of the system, and changing to the continuum, we have

$$\begin{aligned} \frac{D_i(\{N_j\})}{L} &= \frac{N_i}{L} + \frac{G_i^0}{L} - \frac{1}{L} - \int_{-\infty}^{\infty} g(k_i, k') \rho(k') dk' \\ &= \frac{G_i^0 - 1}{L} - \int_{-\infty}^{\infty} (g(k_i, k') - \delta(k_i - k')) \rho(k') dk' \end{aligned} \quad (4.8)$$

We now note that $D_i(\{N_j\})/L$ is the number of particle states per unit length of species i which has $k = k_i$, and so this quantity is equivalent to $(\rho + \rho^h)(k_i)$, with

$$(\rho + \rho^h)(k_i) = \frac{G_i^0 - 1}{L} - \int_{-\infty}^{\infty} (g(k_i, k') - \delta(k_i - k')) \rho(k') dk' \quad (4.9)$$

We now equate equations 4.6 and 4.9, arriving at

$$\begin{aligned} \frac{G_i^0 - 1}{L} &\equiv \frac{1}{2\pi} \\ g(k, k') &= \delta(k - k') - \frac{1}{\pi} \frac{c}{c^2 + (k - k')^2} \end{aligned} \quad (4.10)$$

We note that the first of these equations differs slightly from Wu and Bernard's paper, which used slightly different definitions.

Thus, the spinless boson may be thought of as an ideal exclusion gas, that is, a gas of particles with a statistical interaction but no dynamical interaction. The exclusion statistics are based upon different momenta belonging to different species, with the statistical interaction varying between different species.

We may calculate the energy of our system as in Chapter 3, namely

$$E/L = \int_{-\infty}^{\infty} \rho(k) \epsilon^0(k) dk \quad (4.11)$$

where ϵ_0 is the dispersion relation. This result highlights that there is no interaction energy between particles of different momenta. Of most importance, however, is that the entropy calculated by $S = k_B \ln W$, where W is the statistical weight of the system calculated similarly to above, is the same as the entropy calculated by Yang and Yang [28] with the correspondence $N_i/L \equiv \rho(k)$ and $d_i(\{N_j\}) \equiv \rho^h(k)$. This highlights that any thermodynamics calculated from the statistical interaction approach will be identical to that calculated through the TBA approach.

4.4 FES of the Spinor Boson

Given the success of applying fractional exclusion statistics to the spinless boson and the implications for the calculation of thermodynamics from statistical interactions rather than from the coupled integral equations of the thermodynamic Bethe ansatz method, the next question is whether or not the method may be extended to the spinor boson.

Let us for the moment assume in the Bethe ansatz equations of the spinor boson that all of the spin rapidities are real. Then we may use the integral Bethe ansatz equations in the low temperature approximation. Following the same method as used by Wu and Bernard, it may then be shown that the statistical interaction becomes a 2×2 matrix, where we have

$$\begin{aligned} g_{\mu\nu}(k, k') &= \delta_{\mu\nu}\delta(k, k') + \frac{1}{2\pi}\Omega_{\mu\nu}(k, k') \\ \Omega_{\mu\nu}(k, k') &= -\frac{2c}{c^2 + (k - k')^2} \quad \text{if } \mu = \nu \\ &= \frac{c}{c^2/4 + (k - k')^2} \quad \text{if } \mu \neq \nu \end{aligned} \quad (4.12)$$

Index 0 refers to charge particles, while index 1 refers to spin particles. Thus, the statistical interaction for charge particles is identical to previously, but there also exists a further statistical interaction between charge and spin particles, as well as between spin particles. Note that the the corresponding $(G_i^0 - 1)/L$ present for charge particles is zero for the spin equation.

Hatsugai *et al.* [30] have published an excellent paper in which they comprehensively generically derive this form of the statistical interaction for all Bethe ansatz solvable models. Furthermore, they continued to derive the TBA equations for any generic Bethe ansatz model. However, they do acknowledge that the introduction of multiple levels to the Bethe ansatz equation will inevitably lead to complex rapidities of some form. They postulate that the creation of complex spin rapidities involves significantly more energy than the creation of real spin rapidities, and thus that the equations that they have generically derived are good approximations at low temperature, and are correct at $T = 0$. The rest of the paper deals with specific models in which the energy and thermodynamics are calculated using this approach, and compared to results obtained by other means. All of the models presented corroborate the validity of this approximation.

It is of theoretical interest to find a statistical interaction for the spinor boson, which would allow all interactions to be thought of as purely statistical ones, with no inter-particle interactions. Should a statistical interaction be found, hopefully one could derive thermodynamic properties from a statistical approach, using at the very least a regime in which statistical interactions are approximately constant and may be analysed using methods from the papers presented in Section 2 above. Although it is evident that the method used by Wu and Bernard will not cope with the introduction of complex rapidities, it may be possible to reformulate the approach using a statistical interaction that forms an $N \times N$ matrix which is increased to very large N to account for strings of length N . We discuss this approach in Chapter 7.

4.5 Summary

In this chapter, we introduced the idea of fractional exclusion statistics, in which particles do not obey Bose-Einstein or Fermi-Dirac statistics, but rather, somewhere in-between. A particular form of fractional exclusion statistics known as Haldane exclusion statistics occurs when the addition or removal of a particle of one species affects the number of states left for particles of all other species in a linear manner, with the coefficients for this relation known as the statistical interaction. Non-mutual statistics occur when particles of a species only affect particles of that same species. In the case of non-mutual statistics, thermodynamic information can be derived for the system at low temperature. It has been shown that the spinless Bose gas obeys Haldane exclusion statistics, and thus the question of whether the spinor Bose gas also obeys Haldane exclusion statistics has been raised.

In Chapter 6 we take the approach suggested by Hatsugai *et al.*, and use the TBA in the low temperature approximation to derive thermodynamical quantities for the spinor Bose gas. As of yet, there are no other expressions derived from other methods that we may compare the results to, except for limiting cases. First, however, we derive results for these limiting cases in Chapter 5.

Limiting Regimes

Given the complexity of the continuous Bethe ansatz equations and the thermodynamic Bethe ansatz, both using the string hypothesis and in the low temperature approximation, it is not surprising that no general solution has been found. To simplify the problem, partial solutions have been found in a number of limiting regimes, such as $c = 0$, $c = \infty$, $T = 0$ and $T = \infty$.

$c = 0$ is known as the weak coupling limit, as the interaction between individual particles becomes increasingly small, until we are left with free bosons with spin. At the other end of the scale, $c = \infty$ is known as the strong coupling limit, as the particles become increasingly repulsive. In the limit, particles will not overlap, and so we have what are effectively free fermions. Furthermore, the charge and spin components of the system completely decouple in the strong coupling limit, and so the particles behave as free spinless fermions, and the results from the spinless Bose gas in the strong coupling limit apply.

$c = \infty$ and $T = 0$ are of most interest to this thesis, and are given a thorough investigation. We present the analysis from Gu *et al.* [29] for the strong coupling limit, and our own derivations of previously known results for the zero temperature regime. We give a cursory inspection to the other regimes.

5.1 $c = 0$

While the weak coupling limit corresponds to free bosons for the spinless Bose gas, for the spinor Bose gas, the spin-spin interactions are at their strongest in this limit, and so there is not an exact correspondence between the two models. Gu *et al.* [29] show that Bose-Einstein Condensation does not occur in this system, but otherwise present little of importance in this limit.

5.2 $c = \infty$

At $c = \infty$, known as the strong coupling limit, spin transportation is frozen, and the charge component of the system completely decouples from the spin component. Thus, in this limit, results derived from the charge component will be exactly the same as results for the spinless Bose gas (in the same limit). With regards to the actual physics, the repulsive interaction is so strong that particles do not overlap, and so they behave as noninteracting spinless fermions. Here we present the analysis from Gu *et al.* [29].

In this limit, the TBA equations simplify greatly, leaving only $T \ln(\kappa) = \epsilon(k) = k^2 -$

$\Omega - \mu$. In this simplified form, the integral for the Helmholtz free energy becomes

$$F = \mu N - \frac{TL}{2\pi} \int_{-\infty}^{\infty} \ln(1 + e^{-(k^2 - \Omega - \mu)/T}) dk \quad (5.1)$$

To evaluate the integral, we start by integrating by parts, finding that the boundary term vanishes by way of L'Hôpital's rule. Thus

$$F = \mu N - \frac{L}{\pi} \int_{-\infty}^{\infty} \frac{k^2}{1 + e^{(k^2 - \Omega - \mu)/T}} dk \quad (5.2)$$

The next step is to realise that the integral has an even integrand, and write it as twice the integral from zero to infinity, before making the substitution $u = k^2/T$, to give

$$F = \mu N - \frac{LT^{3/2}}{\pi} \int_0^{\infty} \frac{\sqrt{u}}{1 + e^{u - (\Omega + \mu)/T}} du \quad (5.3)$$

This is now in a form in which it may be approximated for low temperature using Sommerfeld's Lemma (see Appendix A). We find

$$\begin{aligned} \frac{F}{L} &\approx \mu \frac{N}{L} - \frac{T^{3/2}}{\pi} \frac{2}{3} ((\Omega + \mu)/T)^{3/2} \left[1 + \frac{3\pi^2}{24} \frac{1}{((\Omega + \mu)/T)^2} + \dots \right] \\ &= \mu \frac{N}{L} - \frac{2}{\pi} \left[\frac{(\Omega + \mu)^{3/2}}{3} + \frac{\pi^2 T^2}{24(\Omega + \mu)^{1/2}} + \dots \right] \end{aligned} \quad (5.4)$$

To further analyse this, we need to know the chemical potential, and how it varies with temperature. Noting that $\lim_{c \rightarrow \infty} K_n(k) = 0$, we find the following from the integral Bethe ansatz equations (equation 3.24).

$$\begin{aligned} \rho(k)(1 + \kappa) &= \frac{1}{2\pi} \\ \rho(k) &= \frac{1}{2\pi(1 + \kappa)} \\ &= \frac{1}{2\pi} \frac{1}{1 + e^{(k^2 - \mu - \Omega)/T}} \end{aligned} \quad (5.5)$$

We may now use the relation $D = N/L = \int \rho(k) dk$ to determine the chemical potential, by performing the integral, and then solving for μ .

$$\begin{aligned} D &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + e^{(k^2 - \mu - \Omega)/T}} dk \\ &= \frac{\sqrt{T}}{2\pi} \int_0^{\infty} \frac{u^{-1/2}}{1 + e^{u - (\mu + \Omega)/T}} du \end{aligned} \quad (5.6)$$

The substitution $u = k^2/T$ has been made. We apply Sommerfeld's Lemma once again (see Appendix A) to give

$$\begin{aligned} D &\approx \frac{\sqrt{T}}{2\pi} 2 \left(\frac{\mu + \Omega}{T} \right)^{1/2} \left[1 - \frac{\pi^2}{24} \frac{T^2}{(\mu + \Omega)^2} + \dots \right] \\ &= \frac{1}{\pi} (\mu + \Omega)^{1/2} \left[1 - \frac{\pi^2}{24} \frac{T^2}{(\mu + \Omega)^2} + \dots \right] \end{aligned} \quad (5.7)$$

We curtail this at the first order term, as we are working in a low T regime. Squaring both sides and using algebraic manipulation then gives

$$\begin{aligned}\pi^2 D^2 &= (\mu + \Omega) \left[1 - \frac{\pi^2 T^2}{24 (\mu + \Omega)^2} \right]^2 \\ \mu + \Omega &= \frac{\pi^2 D^2}{\left[1 - \frac{\pi^2 T^2}{24 (\mu + \Omega)^2} \right]^2}\end{aligned}\quad (5.8)$$

We find that at $T = 0$, $(\mu + \Omega)_0 = \pi^2 D^2$. Given that we cannot actually solve for μ , we approximate (for low temperature, in which this expression is valid):

$$\begin{aligned}\mu + \Omega &\approx \frac{\pi^2 D^2}{\left[1 - \frac{\pi^2 T^2}{24 (\mu + \Omega)_0^2} \right]^2} \\ &\approx \pi^2 D^2 \left[1 + \frac{T^2}{12\pi^2 D^4} + \dots \right]\end{aligned}\quad (5.9)$$

We now have a temperature dependent chemical potential.

Combining this equation with our expression for the free energy, we arrive at the free energy of the system as a function of temperature, to order T^2

$$\begin{aligned}\frac{F}{L} &\approx \pi^2 D^3 \left[1 + \frac{T^2}{12\pi^2 D^4} \right] - \Omega D - \frac{2}{\pi} \left[\frac{(\pi^2 D^2 \left[1 + \frac{T^2}{12\pi^2 D^4} \right])^{3/2}}{3} + \frac{\pi^2 T^2}{24(\pi^2 D^2)^{1/2}} \right] \\ &= \frac{\pi^2 D^3}{3} - \Omega D - \frac{T^2}{12D} + \mathcal{O}(T^4)\end{aligned}\quad (5.10)$$

Following conventional thermodynamics, (see equation 3.56) we have

$$\frac{S}{L} = \frac{C_v}{L} = \frac{T}{6D} + \mathcal{O}(T^3)\quad (5.11)$$

which is Fermi-liquid like. As required, this is the same as the result for the spinless Bose gas for $c = \infty$.

It is difficult to obtain corrections of order $1/c$ for a strong coupling case, as the isospin and charge do not completely decouple. However, for the spinless case, Batchelor *et al.* [35] have derived a $1/c$ correction to the heat capacity:

$$\frac{C_v}{L} = \frac{T}{6D} \left(1 + \frac{4D}{c} \right) + \mathcal{O}(T^3)\quad (5.12)$$

If the heat capacity for the spinor Bose gas is found, it should be equivalent to this result in the limit $M/N \rightarrow 0$. This topic will be discussed in more detail in Chapter 6.

5.3 $T = 0$

In the $T = 0$ regime, the TBA equations change dramatically. In particular, the first TBA equation becomes

$$\epsilon^-(k) = k^2 - \Omega - \mu + K_2(k) * \epsilon^-(k) + \sum_n K_n(k) * \xi_n^-(k) \quad \text{for } -k_f < k < k_f \quad (5.13)$$

and the free energy (which is here equivalent to the internal energy) becomes

$$F = \mu N - \frac{L}{2\pi} \int_{-k_f}^{k_f} \epsilon(k) dk \quad (5.14)$$

We call k_f the Fermi surface, as there are no particles above this momentum value. Note that ϵ^- and ξ_n^- are always non-positive, as any positive value is lost when setting $T = 0$. The whole integral doesn't vanish because of the $T \ln(1 + e^{-\epsilon(k)/T})$ integrand, which must be carefully treated for negative ϵ and ξ_n^- .

As there will be no holes beneath the Fermi surface, $\kappa(k) = \rho^h/\rho = 0$ for $-k_f < k < k_f$. This means that for no external field, $\ln(\eta_n) = \infty \forall n$, and so there are no spin roots in the system, and thus no spin- \downarrow particles. This means that the ground state is ferromagnetic.

If we start applying an external field (using $\Omega \neq 0$), we find that the ground state may occur for different magnetisations (which corresponds to different M/N). We note that the low temperature approximation is valid here, and assume that all spin rapidities are real. However, instead of using the TBA equations to treat this, we use the integral BAE (equations 3.24).

We now present the results for the spinless Bose gas, followed by the spinor Bose gas with $M = 1$ and $M/N = \frac{1}{2}$.

5.3.1 $M = 0$

This corresponds to the ground state of the spinless Bose gas, which was initially presented and thoroughly explored by Lieb and Liniger [6]. This is a ferromagnetic state, as all particles are in the spin- \uparrow state. It also happens to be the ground state of the system, as seen above. We present only a basic treatment here.

The integral BAE for this system simplifies to

$$(\rho + \rho^h)(k) = \frac{1}{2\pi} + K_2(k) * \rho(k) \quad (5.15)$$

Given that the kinetic energy of the system has the form $E_k/L = \int_{-\infty}^{\infty} k^2 \rho(k) dk$, we expect that ρ will be nonzero only for $-k_f \leq k \leq k_f$, where k_f is the Fermi surface. Below this Fermi surface, there will be no holes, as if there were any, there would be excess energy in the system, and so we wouldn't actually have $T = 0$. Thus we can rewrite our equation as

$$\rho(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-k_f}^{k_f} \frac{2c}{c^2 + (k - k')^2} \rho(k') dk' \quad (5.16)$$

This is an inhomogeneous Fredholm integral equation of the second kind. While an exact solution may be written as the limit of an infinite sum (the Neumann series solution), we may rapidly obtain approximations for $c \ll 1$ and $c \gg 1$.

Limit: $c \gg 1$

We start by Taylor expanding $K_2(k - k')$ for small k' , which is valid because we only have a small range of integration, compared to c , which may be quite large. This gives

$$\rho(k) = \frac{1}{2\pi} + \int_{-k_f}^{k_f} \left(\frac{c}{\pi} \frac{1}{c^2 + k^2} + \frac{2ckk'}{\pi(c^2 + k^2)^2} + \dots \right) \rho(k') dk' \quad (5.17)$$

The first term integrates to give the particle density of the system. The second term integrates to a constant multiplying the total momentum of the system, which we may take to be zero. Subsequent terms are $\mathcal{O}(c^{-3})$, and are thus ignored.

This means that the density of particles within the system is almost constant for large c , with

$$\begin{aligned}\rho(k) &= \frac{1}{2\pi} + \frac{c}{\pi} \frac{1}{c^2 + k^2} \frac{N}{L} \\ &\approx \frac{1}{2\pi} + \frac{1}{\pi c} \frac{N}{L}\end{aligned}\tag{5.18}$$

Note that we have used $k \ll c$ in this approximation. From integrating ρ from $-k_f$ to k_f , we may approximately obtain the Fermi surface for this regime.

$$\begin{aligned}k_f &= \frac{\pi D}{1 + \frac{2D}{c}} \\ &\approx \pi D \left(1 - \frac{2D}{c}\right)\end{aligned}\tag{5.19}$$

As $c \rightarrow \infty$, $k_f = \pi D$ is obtained, which corresponds to the Fermi surface derived above for the $c = \infty$ regime, as required.

Limit: $c \ll 1$

In this limit, we cannot similarly expand $K_2(k - k')$ for small c . Doing so results in $\int_{-k_f}^{k_f} 1/(k - k')^2 dk'$, which is indeterminate, as $-k_f \leq k \leq k_f$. An approximate solution is obtained using the so-called ‘‘semicircle approximation’’, details of which are available in Refs. [36, 37].

$$\begin{aligned}\rho(k) &= \frac{1}{2\pi c} \left(k_f^2 - k^2\right)^{1/2} \\ k_f &= 2\sqrt{cN/L}\end{aligned}\tag{5.20}$$

Evidently, for $c = 0$, the density function diverges, as all particles condense into the ground state, as expected for free bosons.

Figure 5.1 plots these density functions for a variety of c . Note that the red curves for both regimes are for c outside the valid range of the approximations, but give an indication that curves in this region will be some combination of these two.

5.3.2 $M = 1$

It is possible to derive a semi-continuous form of the Bethe ansatz equations, stipulating that the pseudomenta become continuous, whilst keeping only one spin rapidity, and thus having a discrete variable remain. Whilst $M = 1$ is better dealt with by assuming it is distributed out in a continuum and undertaking calculations in this manner, it is theoretically instructive to observe what happens when the value of a single spin rapidity is fixed.

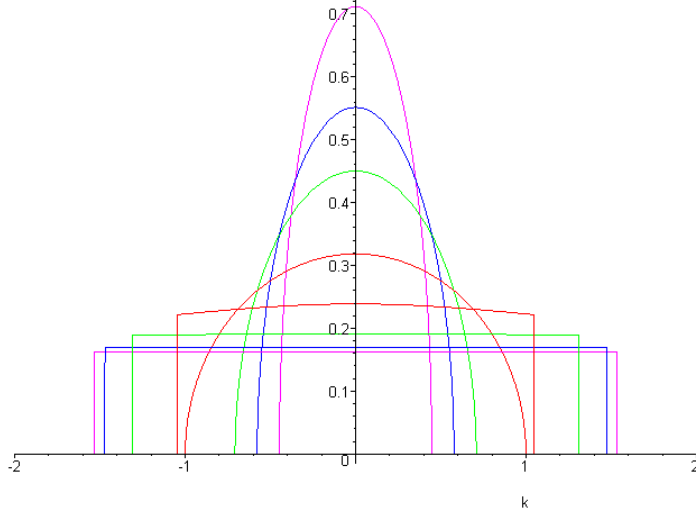


Figure 5.1: Density $\rho(k)$ for varying c at $T = 0$. $D = 0.5$ for all curves. The strongly curved plots are the semicircle approximation, while the plots with sharp edges are of the strong coupling regime. For the semicircle approximation, $c = 1/2$ (Red), $1/4$ (Green), $1/6$ (Blue), $1/10$ (Magenta). For the strong coupling regime, $c = 2$ (Red), 5 (Green), 15 (Blue), 40 (Magenta).

Letting our spin rapidity be called λ , the Bethe ansatz equations then become

$$\begin{aligned} (\rho + \rho^h)(k) &= \frac{1}{2\pi} + K_2(k) * \rho(k) - \frac{1}{2\pi L} \frac{d}{dk} (\Theta_{-1/2}(k - \lambda)) \\ &= \frac{1}{2\pi} + K_2(k) * \rho(k) - \frac{c}{2\pi L} \frac{1}{[c^2/4 + (k - \lambda)^2]} \end{aligned} \quad (5.21)$$

Using this version of the BAE, we can derive density functions for large c in exactly the same manner as above, dropping terms of order c^{-2} as necessary. This new density function is

$$\begin{aligned} \rho(k) &= \frac{1}{2\pi} + \frac{c}{\pi} \frac{1}{c^2 + k^2} \frac{N}{L} - \frac{c}{2\pi L} \frac{1}{[c^2/4 + (k - \lambda)^2]} \quad \text{for } c \gg 1 \\ k_f &= \frac{\pi D}{1 + \frac{2D}{c} - \frac{4}{Lc}} \\ &\approx \pi D \left(1 - \frac{2D}{c} + \frac{4}{Lc}\right) \end{aligned} \quad (5.22)$$

Figure 5.2 illustrates the difference between $M = 0$ and $M = 1$. From the plots, it is evident that a single spin rapidity repels charge roots away from itself, widening the Fermi surface, and thus raising the energy. Note that in the figure, extreme values were chosen to make the effect pronounced. This effect is not limited to the strong coupling case, as Li *et al.* [38] have shown through numerical simulations, and discuss in detail for the $c = \infty$ regime. For more spin rapidities, this effect is compounded. This reinforces that the ground state will be one with no spin rapidities, i.e., where all particles are in same spin state.

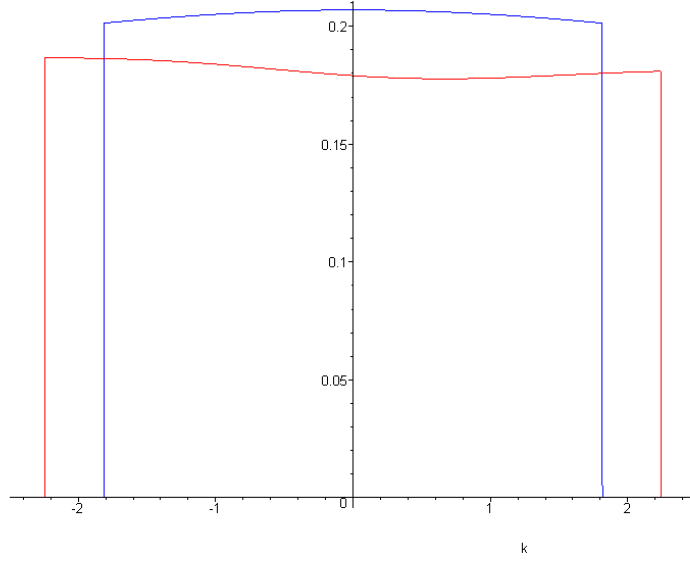


Figure 5.2: Density functions for the strong coupling regime for $M=0$ (Blue) and $M=1$ (Red). The values $c = 5$, $D = 0.8$, $L = 4$, $\lambda = 0.4$ have been used.

5.3.3 Derivations using arbitrary M

We resort to the integral BAE for both charge and spin to solve for arbitrary M . As above, we assume that there are no holes beneath the Fermi surface for the charge, and so we restrict the integrals over charge density to $\pm k_f$, and set ρ^h to zero for this region. However, because the energy of the system is not directly related to the spin density, there is no Fermi surface, and the hole density is not necessarily zero. Except for the special case of $M = N/2$, the system is difficult to solve, requiring the use of the Wiener-Hopf method, which is beyond the scope of this thesis.

Recalling from above that only the negative part of ξ contributes to the system at $T = 0$, we note that for a sufficiently strong external field, all of ξ is negative, and so the distribution of spin roots will be spread out over the whole of λ -space. Furthermore, if we assume that all spin vacancies are occupied by spinon pseudoparticles, i.e., $\sigma^h = 0$, we find that this situation corresponds to a system with $M = N/2$.

Limit: $M = N/2$

We work in the strong coupling regime, and discard all terms of $\mathcal{O}(c^{-2})$ and higher. Thus, for integrals that contain terms of order $1/c$, we use $\rho(k) = 1/2\pi$. The previous results from the strong coupling regime for $M = 0$ then apply. The integral Bethe ansatz equations now read

$$\rho(k) = \frac{1}{2\pi} + \frac{c}{\pi} \frac{1}{c^2 + k^2} \frac{N}{L} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{c}{c^2/4 + (k - \lambda')^2} \sigma(\lambda') d\lambda' \quad (5.23)$$

$$\sigma(\lambda) = \frac{c}{2\pi} \frac{1}{c^2/4 + k^2} \frac{N}{L} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2c}{c^2 + (\lambda - \lambda')^2} \sigma(\lambda') d\lambda' \quad (5.24)$$

We now look at solving the integral equation for $\sigma(\lambda)$, as it is now decoupled. First, we take the Fourier transform of the integral equation. Given that the integral term is a

convolution, the transform is relatively simple to take, with result

$$\hat{\sigma}(w) = \frac{N}{L} e^{-c|w|/2} - \sigma(\hat{w}) e^{-c|w|}$$

From here, it is possible to solve for $\hat{\sigma}(w) = N \operatorname{sech}(cw/2)/2L$. Through the use of residue theory, the inverse Fourier transform may be applied. We find

$$\sigma(\lambda) = \frac{N}{L} \frac{1}{2c} \operatorname{sech}\left(\frac{\pi\lambda}{c}\right)$$

The ratio M/L may now be discovered for $\lambda_B = \infty$ by integrating σ over all λ , with

$$\begin{aligned} \frac{M}{L} &= \int_{-\infty}^{\infty} \sigma(\lambda) d\lambda \\ &= \frac{N}{L} \frac{1}{2c} \int_{-\infty}^{\infty} \operatorname{sech}\left(\frac{\pi\lambda}{c}\right) d\lambda \\ &= \frac{N}{2\pi L} \int_{-\infty}^{\infty} \operatorname{sech}(\lambda) d\lambda \\ &= \frac{N}{2L} \end{aligned} \tag{5.25}$$

Thus, this regime corresponds to half of the particles with spin- \uparrow and half of the particles with spin- \downarrow , as required.

The charge density may now be calculated. First

$$\rho(k) = \frac{1}{2\pi} + \frac{c}{\pi} \frac{1}{c^2 + k^2} \frac{N}{L} - \frac{N}{4\pi L c} \int_{-\infty}^{\infty} \frac{c}{c^2/4 + (k - \lambda')^2} \operatorname{sech}\left(\frac{\pi\lambda'}{c}\right) d\lambda' \tag{5.26}$$

We Taylor expand the integrand for small k . The second order term is an odd function, and thus integrates to zero, and higher order terms are $\mathcal{O}(c^{-5})$, and thus can be ignored.

We have

$$\begin{aligned} \rho(k) &= \frac{1}{2\pi} + \frac{c}{\pi} \frac{1}{c^2 + k^2} \frac{N}{L} - \frac{N}{4\pi L c} \int_{-\infty}^{\infty} \frac{c}{c^2/4 + \lambda'^2} \operatorname{sech}\left(\frac{\pi\lambda'}{c}\right) d\lambda' \\ &= \frac{1}{2\pi} + \frac{c}{\pi} \frac{1}{c^2 + k^2} \frac{N}{L} - \frac{N}{4\pi c L} \int_{-\infty}^{\infty} \frac{\operatorname{sech}(\pi\lambda')}{1/4 + \lambda'^2} d\lambda' \\ &= \frac{1}{2\pi} + \frac{c}{\pi} \frac{1}{c^2 + k^2} \frac{N}{L} - \frac{N \ln(2)}{L \pi c} \\ &= \frac{1}{2\pi} + \frac{1}{\pi c} \frac{N}{L} - \frac{N \ln(2)}{L \pi c} \end{aligned} \tag{5.27}$$

The integral here has been obtained from a comprehensive table of integrals [39, equation 3.522.3], and we have discarded terms of $\mathcal{O}(c^{-2})$.

5.3.4 The Fermionisation of Bosons

We note that in the strong coupling regime, the charge density at zero temperature is dominated by the Fermi surface, which is effectively a step function (see Figure 5.1). It is curious that the particles that we are using, bosons, are displaying a fermion-like structure. This can be attributed to a phenomenon known as the ‘‘Fermionisation of Bosons’’. Because our particles are interacting, Bose-Einstein statistics do not apply. As

the interaction is turned off by $c \rightarrow 0$, the familiar Bose-Einstein statistics re-emerge. As the interaction is made stronger, $c \rightarrow \infty$, the particles behave precisely as fermions, although the wavefunction symmetry is that of bosons. This effect may be explained by considering that the dynamical inter-particle interaction may transmute into a statistical interaction, which becomes Fermi-like in the strong coupling regime. In this manner, the repulsive bosons acquire Fermi-Dirac statistics. Recall that a qualitative illustration of fermionisation was presented in Figure 1.2.

It is important to note that in the strong coupling regime, this Fermi surface structure will dominate low temperature behaviour.

5.3.5 Energies at $T = 0$

Having obtained the charge density for $M = 0$ and $M = N/2$, it is now possible to work out the internal energy per unit length for these limits. We simply integrate $E/L = \int_{-k_f}^{k_f} k^2 \rho(k) dk$ for the charge density in each of the regimes, which is approximately constant. The Fermi surface for each case is easily calculated, as ρ is constant to $\mathcal{O}(c^{-1})$, with

$$k_f = \frac{N/L}{2\rho} \quad (5.28)$$

The energy of each system is simply dependent upon the density, such that

$$\begin{aligned} E/L &= \int_{-k_f}^{k_f} k^2 \rho dk \\ &= \frac{2\rho}{3} [k^3]_0^{k_f} \\ &= \frac{(N/L)^3}{12} \frac{1}{\rho^2} \end{aligned} \quad (5.29)$$

Thus we have

$$\begin{aligned} E/L &= \frac{(N/L)^3}{3} \frac{\pi^2}{\left[1 + \frac{2N}{cL}\right]^2} \quad \text{for } M = 0 \\ E/L &= \frac{(N/L)^3}{3} \frac{\pi^2}{\left[1 + \frac{2N}{cL}(1 - \ln(2))\right]^2} \quad \text{for } M = N/2 \end{aligned} \quad (5.30)$$

Evidently, $M = 0$ has a lower energy than $M = N/2$, and it is a reasonable assumption that intermediate values of M/N interpolate between these two limits in some way. We plot these curves for varying c in Figure 5.3.

To order $\mathcal{O}(c^{-1})$, these equations are

$$\begin{aligned} E/L &= \frac{\pi^2(N/L)^3}{3} - \frac{4\pi^2(N/L)^4}{3c} \quad \text{for } M = 0 \\ E/L &= \frac{\pi^2(N/L)^3}{3} - \frac{4\pi^2(N/L)^4}{3c}(1 - \ln(2)) \quad \text{for } M = \frac{N}{2} \end{aligned} \quad (5.31)$$

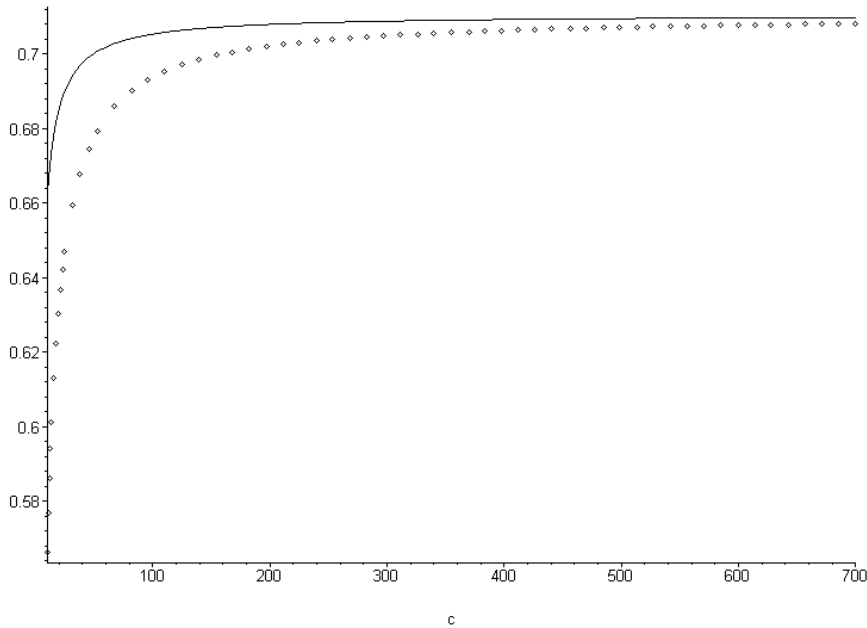


Figure 5.3: Ground state energy for $M = 0$ and $M/N = 1/2$ as a function of c . The solid curve has $M/N = 1/2$, while the dotted curve has $M/N = 0$.

5.4 $T = \infty$

The regime in which $T \rightarrow \infty$ results in free particles and isospins. Gu *et al.* [29] discuss this regime in a small amount of detail, and Takahashi [19, Chapter 12] discusses this limit briefly.

For high T , all $\eta_n(\lambda)$ become independent of λ . The function $\int \ln(1 + e^{-\epsilon(k)/T}) dk$ can then be expanded with respect to the fugacity $e^{-\mu/T}$ to yield the free energy. The free energy at high temperature can also be obtained from fractional exclusion statistics [40].

The most important result in this regime is that the system becomes paramagnetic, with a zero magnetisation.

5.5 Summary

This chapter has dealt primarily with limiting regimes of the model that simplify the TBA equations. The benefit of doing so is twofold; firstly, the model is much simpler to solve in such limits, and secondly, the behaviour of the model in these limits is indicative of how the model will behave in neighbouring regimes.

In the strong coupling limit, the charge and spin components of the system decouple, and so the system is effectively a spinless Bose gas, for which thermodynamical properties were found. In the zero temperature regime, we derived the distribution of charge roots for strong and weak coupling with no spin- \downarrow particles, and showed the effect of a single spin- \downarrow particle on the distributions in the strong coupling limit. We also derived the distribution of charge roots for strong coupling with $M = N/2$, and calculated the ground state energy for strong coupling with $M = 0$ and $M = N/2$.

In the next chapter, we present a new approximation to the TBA equations in the

strong coupling limit for low temperatures. In Chapter 7, we take the analysis of the zero temperature limit with $M = N/2$, and apply fractional exclusion statistics to obtain low temperature behaviour of the system.

A New Approximation

In the previous chapter, it was seen that the integral equations associated with limiting cases of this model are relatively troublesome, with various approximations required to obtain any useful information. It is no surprise that there are no analytical solutions to the thermodynamic Bethe ansatz equations, although a number of attempts have been made to undertake the formidable task of numerically solving the integral equations for certain maximum string lengths. See for example Refs. [41, 42].

In this chapter, we propose a solution to the thermodynamic Bethe ansatz equations in the low temperature approximation (real spin rapidities), as postulated by Hatsugai *et al.* [30]. We work in the strong coupling regime, where $c \rightarrow \infty$, as this is a simpler regime than $c \rightarrow 0$, because at $c = \infty$, the spin components completely decouple from the charge components, as seen in the previous chapter. Furthermore, we already have an analytical expression (equation 5.10) for the free energy at low temperature for the spinless Bose gas in the strong coupling limit, to which we aim to add a $1/c$ correction.

6.1 Low T , Strong Coupling Regime

We start with the TBA equations in the low temperature approximation, which are reproduced here for convenience. We have discarded Ω , the Rabi frequency, to simplify the derivation. Our starting point is thus

$$T \ln(\kappa) = \epsilon(k) = k^2 - \mu - TK_2(k) * \ln(1 + \kappa^{-1}) - TK_1(k) * \ln(1 + \eta^{-1}) \quad (6.1)$$

$$\ln(\eta) = K_1(\lambda) * \ln(1 + \kappa^{-1}) + K_2(\lambda) * \ln(1 + \eta^{-1}) \quad (6.2)$$

Working in the strong coupling regime, $c \rightarrow \infty$, we drop all terms that are $\mathcal{O}(c^{-2})$ and higher. We start by approximating $K_2(k) * \ln(1 + \kappa^{-1})$, by Taylor expanding $K_2(k - k')$ for small k' . Although k' isn't necessarily small, for large k' , the entire function becomes $\mathcal{O}(c^{-2})$, and so we are satisfied that this approximation is valid for the area of interest. Now

$$\begin{aligned} K_2(k) * \ln(1 + \kappa^{-1}) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c}{c^2 + (k - k')^2} \ln(1 + e^{-\epsilon(k')}) dk' \\ &\approx \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{c}{c^2 + k^2} + \frac{2ckk'}{(c^2 + k^2)^2} + \dots \right) \ln(1 + e^{-\epsilon(k')}) dk' \\ &\approx \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c}{c^2 + k^2} \ln(1 + e^{-\epsilon(k')}) dk' \\ &= \frac{2c}{c^2 + k^2} \frac{\mu N - F}{TL} \end{aligned} \quad (6.3)$$

We note that the second order term of the Taylor expansion is $\mathcal{O}(c^{-3})$. The remaining integral is then substituted from equation 3.54, where F is the Helmholtz free energy. We define $J = \frac{\mu N - F}{TL}$ for brevity. $K_1(\lambda) * \ln(1 + \kappa^{-1})$ may be similarly treated, arriving at

$$K_1(\lambda) * \ln(1 + \kappa^{-1}) \approx \frac{c}{c^2/4 + \lambda^2} J \quad (6.4)$$

Making the substitution $\tau = \ln(\eta)$, the second of our TBA equations then becomes

$$\begin{aligned} \tau(\lambda) &= \frac{c}{c^2/4 + \lambda^2} J + \frac{c}{\pi(c^2 + \lambda^2)} * \ln(1 + e^{-\tau(\lambda)}) \\ &= \frac{c}{c^2/4 + \lambda^2} J + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1 + e^{-\tau(\lambda - \lambda')}) d\lambda' \end{aligned} \quad (6.5)$$

Note that while $\ln(\eta) = \xi/T$, including temperature would only mean writing another constant in the equations. Thus, we use $\tau = \xi/T$ instead.

6.2 Solving the TBA Equations

We solve this integral equation by iteration.

$$\begin{aligned} \tau_0 &= \frac{c}{c^2/4 + \lambda^2} J \\ \tau_1 &= \frac{c}{c^2/4 + \lambda^2} J + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1 + e^{-\tau_0(\lambda - \lambda')}) d\lambda' \\ &= \frac{c}{c^2/4 + \lambda^2} J + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1 + e^{-\frac{cJ}{c^2/4 + (\lambda - \lambda')^2}}) d\lambda' \end{aligned} \quad (6.6)$$

Keeping in mind that $\int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} d\lambda' = 1$, we note that for most of the time, the integral term is very close to $\ln(2)$. Figure 6.1 plots the difference between the actual integrand and the integrand using $\ln(2)$ instead of the above formula. As $c \rightarrow \infty$, the integrand converges to $\frac{c}{\pi(c^2 + \lambda'^2)} \ln(2)$.

Thus, we write

$$\begin{aligned} \ln(1 + e^{-\tau(\lambda - \lambda')}) &= \ln(2) - \ln(2) + \ln(1 + e^{-\tau(\lambda - \lambda')}) \\ &= \ln(2) + \ln(1/2 + e^{-\tau(\lambda - \lambda')}/2) \end{aligned} \quad (6.7)$$

where $\ln(2)$ is much bigger than the second term. We may thus write

$$\tau_1 = \frac{c}{c^2/4 + \lambda^2} J + \ln(2) + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1/2 + e^{-\frac{cJ}{c^2/4 + (\lambda - \lambda')^2}}/2) d\lambda' \quad (6.8)$$

For the time being, let us ignore the final term, and look at how the $\ln(2)$ term evolves under iteration. In fact, let's call it $\ln(x)$ for the moment, and find a relation for x which makes this term invariant when iterating.

$$\begin{aligned} \tau_1 &= \frac{c}{c^2/4 + \lambda^2} J + \ln(x) \\ \tau_2 &= \frac{c}{c^2/4 + \lambda^2} J + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1 + e^{-\tau_1(\lambda - \lambda')}) d\lambda' \\ &= \frac{c}{c^2/4 + \lambda^2} J + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1 + e^{-\frac{c}{c^2/4 + (\lambda - \lambda')^2} J - \ln(x)}) d\lambda' \end{aligned}$$

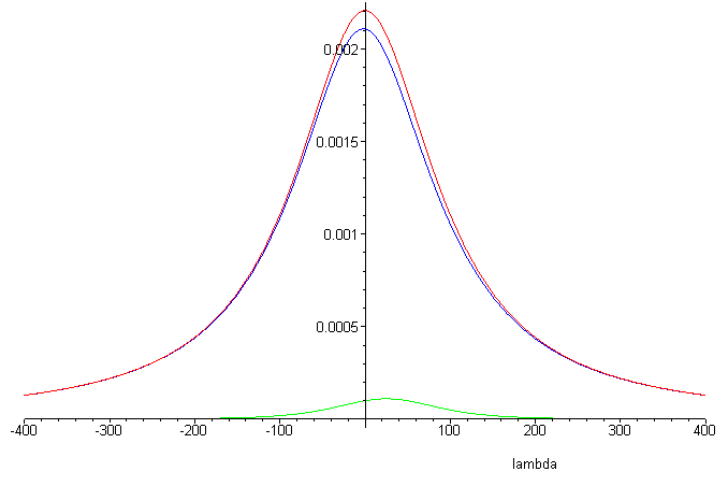


Figure 6.1: An approximation for the integrand for equation 6.6, with $c = 100$, $J = 2$, $\lambda = 50$. The blue line is the correct integrand, the red line is the integrand with a logarithm component of $\ln(2)$, and the green line is the difference of the two.

$$\begin{aligned}
 &= \frac{c}{c^2/4 + \lambda^2} J + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1 + e^{-\frac{c}{c^2/4 + (\lambda - \lambda')^2} J} / x) d\lambda' \\
 &= \frac{c}{c^2/4 + \lambda^2} J + \ln(1 + 1/x) + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln\left(\frac{1 + e^{-\frac{c}{c^2/4 + (\lambda - \lambda')^2} J} / x}{1 + 1/x}\right) d\lambda'
 \end{aligned} \tag{6.9}$$

Thus, we have the condition $\ln(x) = \ln(1 + 1/x)$, or $x = 1 + 1/x$, which is the defining relation for the golden ratio, ϕ . There are two solutions to this equation, one positive, one negative. As we are taking the logarithm of this value, we need the positive one. $\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618\dots$ It is curious to note that no other paper we found mentions the golden ratio. We may thus write our function in the form

$$\tau_2 = \frac{c}{c^2/4 + \lambda^2} J + \ln(\phi) + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln\left(\frac{\phi + e^{-\frac{c}{c^2/4 + (\lambda - \lambda')^2} J}}{\phi + 1}\right) d\lambda' \tag{6.10}$$

We now seek to solve the integral. From the form, it is unlikely that an analytical solution exists. However, we note that if we plot the integrand, the curve looks rather similar to a gaussian for small λ . For larger λ , we may still fit a gaussian to the curve, but the error in the fit increases substantially, before decreasing as $1/c$. As this error is much less than the constant component of η , it is small enough to ignore in this approximation. See Figure 6.2.

We construct the gaussian by centring it at $\lambda = \lambda'$, where the logarithm achieves its maximum value. This is not always the maximum of the curve for large λ , but it is a good estimate. We adjust the height of the gaussian to the height of the curve at that point. Finally, we guess at the width of the gaussian, writing it in terms of c . It appears that there should be no J dependence for this width. A more rigorous method of obtaining

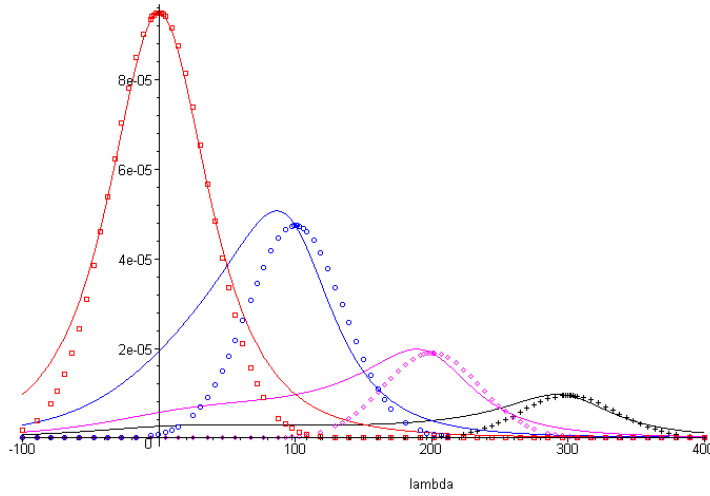


Figure 6.2: Comparison of the function and approximation in equation 6.11. Full lines represent the function, symbols represent the equation. We use $c = 100$, $J = 2$ for all curves. $\lambda = 0$ (Red), 100 (Blue), 200 (Magenta), and 300 (Black).

the width would be to apply curve fitting methods. We thus arrive at the approximation

$$\frac{c}{\pi(c^2 + \lambda^2)} \ln \left(\frac{\phi + e^{-\frac{cJ}{c^2/4 + (\lambda - \lambda')^2}}}{\phi + 1} \right) \approx \frac{c}{\pi(c^2 + \lambda^2)} \ln \left(\frac{\phi + e^{-\frac{4J}{c}}}{\phi + 1} \right) e^{-\frac{4(\lambda - \lambda')^2}{c^2}} \quad (6.11)$$

In this approximation, we may integrate the function, arriving at

$$\tau_2 = \frac{c}{c^2/4 + \lambda^2} J + \ln(\phi) + \frac{c^2}{2\sqrt{\pi}(c^2 + \lambda^2)} \ln \left(\frac{\phi + e^{-\frac{4J}{c}}}{\phi + 1} \right) \quad (6.12)$$

While certainly elegant, this last term is also rather bulky, and may be Taylor expanded in terms of large c . It is interesting to note that this term has no λ dependence until the c^{-3} term. Keeping only the first order term, we have

$$\begin{aligned} \tau_2 &= \frac{c}{c^2/4 + \lambda^2} J + \ln(\phi) - \frac{2J}{c\sqrt{\pi}\phi^2} \\ &= \frac{c}{c^2/4 + \lambda^2} J + \ln(\phi e^{-\frac{2J}{c\sqrt{\pi}\phi^2}}) \end{aligned} \quad (6.13)$$

We note that our constant is now not quite the golden ratio, but a slightly changed version. Assuming our constant is now $\ln(A)$, we find that our equilibrium condition is now:

$$\ln(A) = \ln\left(1 + \frac{1}{A}\right) - \frac{2J}{c\sqrt{\pi}(1 + A)} \quad (6.14)$$

This is a transcendental equation for A . We approximate the solution by exponentiating both sides, before Taylor expanding to $\mathcal{O}(c^{-1})$ and solving for A . We then Taylor expand

again, finding the $1/c$ correction to $\ln(\phi)$. In this way we find

$$\begin{aligned} A &\approx \frac{1 + \sqrt{5 - \frac{8J}{c\sqrt{\pi}}}}{2} \\ &\approx \phi - \frac{2J}{\sqrt{5\pi}c} \\ \ln(A) &= \ln\left(\phi - \frac{2J}{\sqrt{5\pi}c}\right) \\ &\approx \ln(\phi) - \frac{2J}{\sqrt{5\pi}c\phi} \end{aligned} \quad (6.15)$$

Thus, to order $1/c$, we find that

$$\ln(\eta) = \frac{c}{c^2/4 + \lambda^2} J + \ln(\phi) - \frac{2J}{\sqrt{5\pi}c\phi} \quad (6.16)$$

Having obtained η , we now seek to solve for $\epsilon(k)$ via

$$\epsilon(k) = k^2 - \mu - T \frac{2cJ}{c^2 + k^2} - TK_1(k) * \ln(1 + \eta^{-1}) \quad (6.17)$$

The final term can be easily found by following steps similar to the above derivation. We use $\ln(A)$ instead of it's value, as the integral is slightly different. Now

$$\begin{aligned} K_1(k) * \ln(1 + \eta^{-1}) &= \int_{-\infty}^{\infty} \frac{c}{2\pi(c^2/4 + \lambda'^2)} \ln(1 + e^{-\tau(\lambda - \lambda')}) d\lambda' \\ &= \ln\left(1 + \frac{1}{A}\right) - \frac{4J}{c\sqrt{\pi}(1 + A)} \\ &= 2\ln(A) - \ln\left(1 + \frac{1}{A}\right) \\ &= \ln\left(\frac{A^3}{1 + A}\right) \\ &\approx \ln(\phi) - \frac{J}{c\sqrt{5\pi}\phi} \left(6 - \frac{2}{\phi}\right) \end{aligned} \quad (6.18)$$

We have used the approximate value for A from above, and Taylor expanded for small $1/c$. We then have the result

$$\epsilon(k) = k^2 - \mu - \frac{2cJT}{c^2 + k^2} - T\ln(\phi) + \frac{JT}{c\sqrt{5\pi}\phi} \left(6 - \frac{2}{\phi}\right) \quad (6.19)$$

6.3 Calculating the Free Energy

We may now calculate the free energy per unit length, using equation 3.54. We start by integrating by parts, where the boundary terms vanish by way of L'Hôpital's rule.

$$\begin{aligned} F/L &= \mu N/L - \frac{T}{2\pi} \int_{-\infty}^{\infty} \ln(1 + e^{-\epsilon(k)/T}) dk \\ &= \mu N/L - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k}{1 + e^{\epsilon(k)/T}} \frac{\partial \epsilon(k)}{\partial k} dk \end{aligned}$$

$$= \mu N/L - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k}{1 + e^{\epsilon(k)/T}} \left(2k + \frac{4kcJT}{(c^2 + k^2)^2} \right) dk \quad (6.20)$$

The second term in the brackets is $\mathcal{O}(c^{-3})$, compared to the first term which is $\mathcal{O}(1)$, and so we drop the second term. Thus

$$F/L = \mu N/L - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{k^2}{1 + e^{\epsilon(k)/T}} dk \quad (6.21)$$

Next, we note that $\epsilon(k)$ is an even function in k , as is the rest of the integrand, and so we change the limits of integration to $0 \dots \infty$, and multiply by two. This allows us to make the substitution $u = k^2/T$, with result

$$\begin{aligned} F/L &= \mu N/L - \frac{2}{\pi} \int_0^{\infty} \frac{k^2}{1 + e^{\epsilon(k)/T}} dk \\ &= \mu N/L - \frac{1}{\pi} \int_0^{\infty} \frac{u^{1/2} T^{3/2}}{1 + e^{\epsilon(u)/T}} du \\ &= \mu N/L - \frac{1}{\pi} \int_0^{\infty} \frac{u^{1/2} T^{3/2}}{1 + e^{u - \mu/T - \frac{2cJ}{c^2 + Tu} - \ln(\phi) + \frac{J}{c\sqrt{5\pi}\phi} (6 - \frac{2}{\phi})}} du \end{aligned} \quad (6.22)$$

In the exponent, u clearly dominates $\frac{2cJ}{c^2 + Tu}$, and so we Taylor expand this second term for large c , before grouping the terms together. Now

$$\begin{aligned} F/L &= \mu N/L - \frac{T^{3/2}}{\pi} \int_0^{\infty} \frac{u^{1/2}}{1 + e^{u - \left[\mu/T + 2J/c + \ln(\phi) - \frac{J}{c\sqrt{5\pi}\phi} (6 - \frac{2}{\phi}) \right]}} du \\ &= \mu N/L - \frac{T^{3/2}}{\pi} \int_0^{\infty} \frac{u^{1/2}}{1 + e^{u-B}} du \end{aligned} \quad (6.23)$$

We have substituted $B = \mu/T + 2J/c + \ln(\phi) - \frac{J}{c\sqrt{5\pi}\phi} (6 - \frac{2}{\phi})$. This is the form required for Sommerfeld's lemma (see Appendix A), where B is large, as this holds only for low temperature. Applying Sommerfeld's lemma, we arrive at

$$\begin{aligned} F/L &\approx \mu N/L - \frac{2}{3\pi} (TB)^{3/2} \left(1 + \frac{\pi^2}{8} \frac{T^2}{(TB)^2} + \dots \right) \\ &= \mu N/L - \frac{2}{3\pi} (TB)^{3/2} - \frac{\pi}{12} \frac{T^2}{(TB)^{1/2}} + \dots \end{aligned} \quad (6.24)$$

We have $TB = \mu + 2JT/c + T \ln(\phi) - \frac{JT}{c\sqrt{5\pi}\phi} (6 - \frac{2}{\phi})$. Since all other terms have factors of T or $1/c$, μ is the biggest term here. Now take μ out as a factor, before using the binomial expansion to first order to obtain $(TB)^x$, with result

$$\begin{aligned} TB &= \mu \left[1 + \frac{2JT}{\mu c} + \frac{T \ln(\phi)}{\mu} - \frac{JT}{\mu c\sqrt{5\pi}\phi} (6 - \frac{2}{\phi}) \right] \\ (TB)^{3/2} &\approx \mu^{3/2} \left[1 + \frac{3}{2} \frac{2JT}{\mu c} + \frac{3}{2} \frac{T \ln(\phi)}{\mu} - \frac{3}{2} \frac{JT}{\mu c\sqrt{5\pi}\phi} (6 - \frac{2}{\phi}) \right] \\ &= \left[\mu^{3/2} + \frac{3JT\mu^{1/2}}{c} + \frac{3T \ln(\phi)\mu^{1/2}}{2} - \frac{3JT\mu^{1/2}}{c\sqrt{5\pi}\phi} (3 - \frac{1}{\phi}) \right] \end{aligned} \quad (6.25)$$

$$\begin{aligned}
(TB)^{-1/2} &\approx \mu^{-1/2} \left[1 - \frac{JT}{\mu c} - \frac{T \ln(\phi)}{2\mu} + \frac{JT}{\mu c \sqrt{5\pi\phi}} \left(3 - \frac{1}{\phi} \right) \right] \\
&= \left[\mu^{-1/2} - \frac{JT\mu^{-3/2}}{c} - \frac{T \ln(\phi)\mu^{-3/2}}{2} + \frac{JT\mu^{-3/2}}{c\sqrt{5\pi\phi}} \left(3 - \frac{1}{\phi} \right) \right] \quad (6.26)
\end{aligned}$$

Combining all of this together gives

$$\begin{aligned}
F/L &\approx \mu N/L - \frac{2}{3\pi} \left[\mu^{3/2} + \frac{3T \ln(\phi)\mu^{1/2}}{2} + \frac{3JT\mu^{1/2}}{c} \left(1 - \frac{1}{\sqrt{5\pi\phi}} \left(3 - \frac{1}{\phi} \right) \right) \right] \\
&\quad - \frac{\pi T^2}{12} \left[\mu^{-1/2} - \frac{T \ln(\phi)\mu^{-3/2}}{2} - \frac{JT\mu^{-3/2}}{c} \left(1 - \frac{1}{\sqrt{5\pi\phi}} \left(3 - \frac{1}{\phi} \right) \right) \right] \quad (6.27)
\end{aligned}$$

Now, recall that $J = \frac{\mu N - F}{TL}$, and so we now need to solve for F/L . For brevity, let us denote $\left(1 - \frac{1}{\sqrt{5\pi\phi}} \left(3 - \frac{1}{\phi} \right) \right)$ by $\chi \approx 0.629$.

$$\begin{aligned}
F/L &\approx \mu N/L - \frac{2}{3\pi} \left[\mu^{3/2} + \frac{3T \ln(\phi)\mu^{1/2}}{2} + \frac{\mu N - F}{L} \frac{3\mu^{1/2}}{c} \chi \right] \\
&\quad - \frac{\pi T^2}{12} \left[\mu^{-1/2} - \frac{T \ln(\phi)\mu^{-3/2}}{2} - \frac{\mu N - F}{L} \frac{\mu^{-3/2}}{c} \chi \right] \\
&= \mu N/L - \frac{2}{3\pi} \left[\mu^{3/2} + \frac{3T \ln(\phi)\mu^{1/2}}{2} + \frac{\mu N}{L} \frac{3\mu^{1/2}}{c} \chi \right] + \frac{F}{L} \frac{2\mu^{1/2}}{\pi c} \chi \\
&\quad - \frac{\pi T^2}{12} \left[\mu^{-1/2} - \frac{T \ln(\phi)\mu^{-3/2}}{2} - \frac{\mu N}{L} \frac{\mu^{-3/2}}{c} \chi \right] - \frac{F}{L} \frac{\pi T^2 \mu^{-3/2}}{12c} \chi
\end{aligned}$$

$$\begin{aligned}
\frac{F}{L} \left(1 - \frac{2\mu^{1/2}}{\pi c} \chi + \frac{\pi T^2 \mu^{-3/2}}{12c} \chi \right) &= \mu \frac{N}{L} - \frac{2}{3\pi} \mu^{3/2} \left(1 + \frac{\pi^2 T^2}{8\mu^2} \right) \\
&\quad - \left(\frac{T \ln(\phi)\mu^{1/2}}{\pi} + \frac{N}{L} \frac{2\mu^{3/2}\chi}{\pi c} \right) \left(1 - \frac{\pi^2 T^2}{24\mu^2} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{F}{L} &\approx \left[\mu \frac{N}{L} - \frac{2}{3\pi} \mu^{3/2} \left(1 + \frac{\pi^2 T^2}{8\mu^2} \right) \right. \\
&\quad \left. - \left(\frac{T \ln(\phi)\mu^{1/2}}{\pi} + \frac{N}{L} \frac{2\mu^{3/2}\chi}{\pi c} \right) \left(1 - \frac{\pi^2 T^2}{24\mu^2} \right) \right] \left[1 + \frac{2\mu^{1/2}}{\pi c} \chi - \frac{\pi T^2 \mu^{-3/2}}{12c} \chi + \dots \right] \\
&= \mu \frac{N}{L} - \frac{2}{3\pi} \mu^{3/2} \left(1 + \frac{\pi^2 T^2}{8\mu^2} \right) - \frac{T \ln(\phi)\mu^{1/2}}{\pi} - \frac{N}{L} \frac{2\mu^{3/2}\chi}{\pi c} \left(1 - \frac{\pi^2 T^2}{24\mu^2} \right) \\
&\quad + \frac{2\mu^{1/2}}{\pi c} \chi \left[\mu \frac{N}{L} - \frac{2}{3\pi} \mu^{3/2} \left(1 + \frac{\pi^2 T^2}{8\mu^2} \right) - \frac{T \ln(\phi)\mu^{1/2}}{\pi} \right] \\
&\quad - \frac{\pi T^2 \mu^{-3/2}}{12c} \chi \left[\mu \frac{N}{L} - \frac{2}{3\pi} \mu^{3/2} \right] \\
&= \mu \frac{N}{L} - \frac{2}{3\pi} \mu^{3/2} \left(1 + \frac{\pi^2 T^2}{8\mu^2} \right) - \frac{T \ln(\phi)\mu^{1/2}}{\pi} - \frac{4}{3\pi^2 c} \mu^2 \chi - \frac{2T \ln(\phi)\mu}{\pi^2 c} \chi - \frac{T^2}{9c} \chi \quad (6.28)
\end{aligned}$$

This is the most significant result of this approximation. From here, all of the further results are derived.

At $T = 0$, the free energy simplifies to

$$F/L = \mu N/L - \frac{2\mu^{3/2}}{3\pi} - \frac{4\mu^2}{3\pi^2 c} \chi \quad (6.29)$$

We now need the chemical potential at zero temperature. Let us start with a trial value of $\mu = \pi^2 D^2 (1 + \alpha/c)$. We know that the energy for zero temperature is of the form $E/L = \pi^2 D^2/3 - 4\pi^2 D^4/3c \times (1 - x)$, and so we check that this trial value works by equating this with the derived value for F/L , using the binomial expansion and dropping all terms in c^{-2} . Explicitly

$$\begin{aligned} \frac{\pi^2 D^3}{3} - \frac{4\pi^2 D^4}{3c} (1 - x) &= \mu D - \frac{2\mu^{3/2}}{3\pi} - \frac{4\mu^2}{3\pi^2 c} \chi \\ &= \pi^2 D^3 (1 + \alpha/c) - \frac{2}{3} \pi^2 D^3 (1 + \alpha/c)^{3/2} \\ &\quad - \frac{4\pi^2 D^4}{3c} \chi (1 + \alpha/c)^2 \\ &\approx \pi^2 D^3 (1 + \alpha/c) - \frac{2}{3} \pi^2 D^3 (1 + \frac{3\alpha}{2c}) \\ &\quad - \frac{4\pi^2 D^4}{3c} \chi (1 + 2\alpha/c) \\ &\approx \frac{\pi^2 D^3}{3} - \frac{4\pi^2 D^4}{3c} \chi \end{aligned} \quad (6.30)$$

Thus, this trial value satisfies the form of the energy required. However, this gave no insight into the value of α . As we are unable to calculate the Fermi surface in this regime (as the charge and spin components have not completely decoupled, the integral equations are difficult to solve), we turn to the thermodynamic formula $\frac{\partial F}{\partial N} = \mu$. Note that we must treat μ as a function of particle number. The result

$$\frac{\partial F}{\partial N} = \mu + N \frac{\partial \mu}{\partial N} - \frac{L\mu^{1/2}}{\pi} \frac{\partial \mu}{\partial N} - \frac{8L\mu}{3\pi^2 c} \chi \frac{\partial \mu}{\partial N} \quad (6.31)$$

implies that

$$\frac{\partial \mu}{\partial N} \left(N - \frac{L\mu^{1/2}}{\pi} - \frac{8L\mu}{3\pi^2 c} \chi \right) = 0 \quad (6.32)$$

As the trial form of μ is a function of N , then $\partial \mu / \partial N \neq 0$, and so

$$N - \frac{L\mu^{1/2}}{\pi} - \frac{8L\mu}{3\pi^2 c} \chi = 0 \quad (6.33)$$

We insert the trial value for μ , drop all terms in c^{-2} , and solve for α to find

$$\begin{aligned} N - N(1 + \alpha/c)^{1/2} - \frac{8ND}{3c} \chi &= 0 \\ 1 - (1 + \frac{\alpha}{2c}) - \frac{8D}{3c} \chi &= 0 \\ \alpha &= -\frac{16D}{3} \chi \end{aligned} \quad (6.34)$$

We then have the chemical potential at $T = 0$:

$$\mu = \pi^2 D^2 \left(1 - \frac{16D\chi}{3c} \right) \quad (6.35)$$

Unfortunately, the method previously used to calculate the chemical potential (Section 5.2) can't be applied, as the density function is no longer as trivial as in the strong coupling limit. However, as there is no simple way to obtain the temperature dependence of μ , we approximate that the previously derived temperature dependence is the same. This is a reasonable approximation, as it will give the correct result in the limiting cases of $T = 0$ and $c = \infty$. Furthermore, the correction to the chemical potential is of order $1/c$, which is small regardless. The correct way to calculate the chemical potential is to solve for the charge density, and integrate it over all k to obtain the particle density, then solve for μ . However, the solution to the coupled integral equations required to solve for the charge density is beyond the scope of this thesis.

The approximation that we will use for the chemical potential is (cf equation 5.9)

$$\begin{aligned} \mu &= \frac{\mu_0}{\left[1 - \frac{\pi^2 T^2}{24 \mu_0^2} \right]^2} \\ &\approx \mu_0 \left[1 + \frac{\pi^2 T^2}{24 \mu_0^2} + \dots \right] \\ &\approx \pi^2 D^2 \left(1 - \frac{16D\chi}{3c} \right) \left[1 + \frac{1}{24} \frac{T^2}{\pi^2 D^4} \left(1 + \frac{32D\chi}{3c} \right) \right] \end{aligned} \quad (6.36)$$

We do not present the full expression for the free energy as a function of temperature, as inserting the chemical potential does not elucidate anything. The heat capacity of the system, calculated using eqns 3.56, may be expanded to order T^2 and c^{-1} . We arrive at the result

$$\frac{C_v}{L} \approx \frac{T}{6D} \left(1 + \frac{4D\chi}{c} \right) + \frac{T^2 \ln(\phi)}{\pi^2 D^2} \left(\frac{1}{8D} + \frac{3\chi}{2c} \right) \quad (6.37)$$

6.4 Comparisons to Other Regimes

6.4.1 $c = \infty$

In the strong coupling limit, using $\mu = \pi^2 D^2 \left[1 + \frac{T^2}{12\pi^2 D^4} \right]$, we have the free energy

$$\frac{F}{L} = \frac{\pi^2 D^3}{3} - \frac{T^2}{12D} - TD \ln(\phi) \left[1 + \frac{T^2}{24\pi^2 D^4} \right] \quad (6.38)$$

We now compare this to equation 5.10, and note that we have obtained an extra $\ln(\phi)$ term.

6.4.2 $T = 0$

At $T = 0$, the ground state is ferromagnetic as we have discarded the Rabi frequency term, and so we should recover equation 5.31 for $M = 0$. Using the chemical potential derived above, we find that

$$\frac{F}{L} = \frac{\pi^2 D^3}{3} - \frac{4\pi^2 D^4}{3c} \chi \quad (6.39)$$

In this case, unless $\chi = 1$ (which by definition, it isn't), this equation is different to the previously calculated value.

6.5 Explaining the Discrepancies

All of the discrepancies that exist between our results and the limiting cases can be seen to stem from the final term in $\epsilon(k)$, which gives rise to χ and $\ln(\phi)$. Unable to find errors in our method, we carried out a careful inspection of the limiting case, $c = \infty$.

Let us look at the second of the TBA equations in the strong coupling regime, with no external field, namely

$$\begin{aligned} \tau(\lambda) &= \frac{c}{c^2/4 + \lambda^2} J + K_2(\lambda) * \ln(1 + e^{-\tau(\lambda)}) \\ &= \frac{c}{c^2/4 + \lambda^2} J + \int_{-\infty}^{\infty} \frac{c}{\pi(c^2 + \lambda'^2)} \ln(1 + e^{-\tau(\lambda - \lambda')}) d\lambda' \end{aligned} \quad (6.40)$$

Make a change of coordinates $x = \lambda/c$, and substitute this in the integral also, giving

$$\tau(\lambda) = \frac{J}{c(1/4 + x^2)} + \int_{-\infty}^{\infty} \frac{1}{\pi(1 + x'^2)} \ln(1 + e^{-\tau(x - x')}) dx' \quad (6.41)$$

We note that as $c \rightarrow \infty$, the first term vanishes, but the second is completely independent of c . Unsurprisingly, the solution to this integral equation is $\tau(x) = \tau(\lambda) = \ln(\phi)$, where ϕ is still the golden ratio. Now, we move on to look at the first TBA equation in the strong coupling regime, which is

$$\begin{aligned} \epsilon(k) &= k^2 - \mu - T \frac{2cJ}{c^2 + k^2} - TK_1(k) * \ln(1 + e^{\tau(k)}) \\ &= k^2 - \mu - T \frac{2cJ}{c^2 + k^2} - T \int_{-\infty}^{\infty} \frac{c}{2\pi(c^2/4 + k'^2)} \ln(1 + e^{-\tau(k - k')}) dk' \end{aligned} \quad (6.42)$$

Again, apply a change of coordinates $\beta = k/c$, giving

$$\begin{aligned} \epsilon(\beta) &= c^2 \beta^2 - \mu - T \frac{2J}{c(1 + \beta^2)} - T \int_{-\infty}^{\infty} \frac{1}{2\pi(1/4 + \beta'^2)} \ln(1 + e^{-\tau(\beta - \beta')}) d\beta' \\ &= c^2 \beta^2 - \mu - T \frac{2J}{c(1 + \beta^2)} - T \int_{-\infty}^{\infty} \frac{1}{2\pi(1/4 + \beta'^2)} \ln(1 + e^{-\ln(\phi)}) d\beta' \\ &= c^2 \beta^2 - \mu - T \frac{2J}{c(1 + \beta^2)} - T \ln(\phi) \end{aligned} \quad (6.43)$$

Changing back to $k = c\beta$, and taking the limit $c \rightarrow \infty$, we arrive at

$$\epsilon(k) = k^2 - \mu - T \ln(\phi) \quad (6.44)$$

This is contrary to the result from the analysis of Gu *et al.* [29] which we presented in the previous chapter. If this new expression for $\epsilon(k)$ is used in the $c = \infty$ analysis, the equations that we have derived in this chapter are correct in the strong coupling limit.

The investigation of this discrepancy is beyond the scope of this thesis, but remains a question of interest.

It is a known feature of the TBA model that the order in which limits are taken is important, including $c \rightarrow \infty$, $T \rightarrow 0$, and also the thermodynamic limit. Mathematically, it is well known that uniform convergence is required to interchange the order in which limits are taken, and so for this model, the limits do not commute. Thus results are dependent upon the order in which limits are taken. It is believed that any further discrepancies between limits cases of these equations arise from this phenomena. In particular, we have made a tacit assumption that $cT \gg 1$, and so equations involving the zero temperature limit are likely to yield different results to what would otherwise be expected. We are thus satisfied with our results.

6.6 Summary

In this chapter, we have presented our own analysis of the model in the strong coupling regime at low temperatures. We used the TBA equations in the low temperature approximation, as suggested by Hatsugai *et al.* [30], and solved them to order $1/c$, calculating the free energy of the system to order T^2 . Next, we calculated the chemical potential for the system at low temperatures, and combined this with the expression for free energy to obtain the heat capacity at constant volume. The significance of these results is that the first order correction to the strong coupling limit contains all of the contributions from spin, and so represents a large step forwards in understanding the strong coupling limit of the spinor Bose gas.

6.7 Further Work

The desired result from this line of enquiry is to obtain a first order correction in c^{-1} to the free energy at low temperatures, and from there to calculate M/N as a function of temperature. This would allow the calculation of the magnetisation $\propto (N - 2M)$ by temperature. This is of interest, as the ground state is known to be ferromagnetic, while at high temperatures, the system is paramagnetic, and relatively little is known of the intermediate regimes.

To obtain M/N , one would take the functions for $\epsilon(k)$ and $\eta(\lambda)$, and substitute them into the BAE in integral form (equations 3.27), eliminating the hole densities from these equations. This leaves two coupled integral equations to solve, both involving temperature dependent terms. Once a solution for $\rho(k)$ and $\sigma(\lambda)$ has been obtained, the relations $N/L = \int \rho(k) dk$ and $M/L = \int \sigma(\lambda) d\lambda$ may be used to obtain the ratio M/N .

An alternative form of the free energy might also be used by deriving the TBA equations using a fixed ratio M/N , allowing the calculation of the free energy for a fixed magnetisation, which may then be minimised with regards to M/N . This should result in a minimisation condition for M/N , which would theoretically be the same as the result obtained from above.

A related problem which arises when dealing with magnetisation is how the magnetisation of the system varies with different external fields Ω . Like ferromagnetic systems, this system will display hysteresis. By considering the BAE and TBA equations at zero temperature for different external fields, the form of this hysteresis curve could potentially be derived. While the critical external field strength needed to create a zero magnetisation is known, the linearised effect of small perturbations to the field about this critical strength on the magnetisation is unknown.

Thermodynamics from Fractional Exclusion Statistics

As mentioned in Chapter 4, since the connection between the spinless Bose gas and Haldane exclusion statistics was identified by Wu and Bernard in 1994 [21], the question of whether multi-component Bethe ansatz solvable models may also be identified as exclusions obeying fractional exclusion statistics has been of considerable theoretical interest. In this chapter, we look at an attempt to construct a statistical interaction for multi-component Bethe ansatz solvable models, and explain how such efforts have to date proved fruitless. We also discuss the usefulness of the identification of a statistical interaction. We found one specific case in which non-mutual statistics may be approximated for the spinor Bose gas, and present this with the thermodynamics that fractional exclusion statistics yields for this case.

7.1 Attempts at Incorporating the String Hypothesis

Similar to the method used by Bernard and Wu [21], we can use the integral Bethe ansatz equations in an attempt to define a statistical interaction $g_{\nu\mu}(\lambda, \lambda')$, with the subscripts referring to charge particles and different length strings of spin rapidities. This would imply mutual exclusion statistics between strings of different lengths, as well as particles and spin rapidity strings. The problem with this approach is that the statistical weight then becomes impossible to derive, as instead of counting the possibilities for individual particles, we now have to consider the possibilities for individual strings containing individual particles. This becomes an intractable problem when strings of infinite length are considered.

7.2 The Significance of the Statistical Interaction

Although the discovery of a statistical interaction for the spinor Bose gas involving complex spin rapidities would be a major achievement should it be derived, but it is unlikely to be of any practical use. The thermodynamic equations derived from such a statistical interaction would be completely equivalent to the TBA equations discussed in Chapter 3, and thus, just as impossible to solve. In this regards, the statistical interaction obeyed by the spinor Bose is essentially of theoretical interest only. However, if such a statistical interaction is found to exist, then the spinor Bose gas may be defined as an exclusion gas obeying fractional exclusion statistics, and comparisons to other exclusion gases may then be drawn.

The exception to this is when non-mutual statistics may be found in particular regimes, as is presented in the next section. In this case, significant results can be found by knowing the statistical interaction.

7.3 A Special Case

In the case of the strong coupling regime at zero temperature, a particularly elegant solution for the spinon root density was found for the case of $M/N = 1/2$ in Chapter 5 (Section 5.3.3). The resulting charge and spin densities are reproduced here for convenience.

$$\begin{aligned}\rho(k) &= \frac{1}{2\pi} + \frac{1}{\pi c} \frac{N}{L} (1 - \ln(2)) \\ \sigma(\lambda) &= \frac{N}{L} \frac{1}{2c} \operatorname{sech}\left(\frac{\pi\lambda}{c}\right)\end{aligned}\quad (7.1)$$

Although the equations were originally coupled, the form in which they are presented here are decoupled. In this manner, non-mutual statistics now apply, $g_{\mu\nu} = \delta_{\mu\nu}g_{\mu}$, indicating that the charge and spin components are (effectively) decoupled. Since the charge component dictates the energy of the system, we can essentially ignore the statistical interaction between spinons.

The total number of states in the system is simply $N/L = \int \rho(k) dk$, where the limits of integration are $\pm k_f$, the Fermi surface. As ρ is a constant, we can write this out in full as

$$\frac{N}{L} = 2k_f \left(\frac{1}{2\pi} + \frac{1}{\pi c} \frac{N}{L} (1 - \ln(2)) \right) \quad (7.2)$$

But at the same time, the total number of states can be written as $D_i = N_i + G_i^0 - g_{ii}N_i - 1$, in the same manner as equation 4.7, where we treat all charge particles as belonging to the same species. Because we have non-mutual statistics, there is no sum over the statistical interaction with the spinons. We assume that the statistical interaction is approximately constant. When we equate this expression with the above equation, we find that

$$2k_f \left(\frac{1}{2\pi} + \frac{1}{\pi c} \frac{N}{L} (1 - \ln(2)) \right) = \frac{G^0 - 1}{L} + \frac{N}{L} (1 - g) \quad (7.3)$$

Now, given that when there are no particles in the system, the charge hole density is $1/2\pi$, we find that $(G^0 - 1)/L = k_f/\pi$. Consequently, we have

$$\frac{2k_f}{\pi c} \frac{N}{L} (1 - \ln(2)) = \frac{N}{L} (1 - g) \quad (7.4)$$

Now k_f was calculated in equation 5.28. Solving for g and dropping terms of order c^{-2} , we obtain

$$g = 1 - \frac{2}{c} \frac{N}{L} (1 - \ln(2)) \quad (7.5)$$

In this manner, we have obtained a constant statistical interaction for this particular case.

There is reason to believe that the statistics for low temperature do not differ greatly from the statistics for zero temperature [30], and so we approximate that the statistical interaction is constant for low temperatures. We may now apply the results of Isakov *et al.* [23], with help from Batchelor and Guan [43], who provide formulae for calculating energies from the fractional exclusion statistics formulation. We use $D = \frac{N}{L}$.

The chemical potential at $T = 0$ calculated by fractional exclusion statistics is

$$\begin{aligned}\mu_0 &= \pi^2 D^2 g^2 \\ &= \pi^2 D^2 \left(1 - \frac{4D}{c}(1 - \ln(2))\right)\end{aligned}\quad (7.6)$$

which is exactly the same as when calculated by $\mu_0 = k_f^2$. The chemical potential at low temperatures is then found

$$\begin{aligned}\mu &= \mu_0 \left[1 + \frac{\pi^2 T^2}{12\mu_0^2} g\right] \\ &= \mu_0 \left[1 + \frac{\pi^2 T^2}{12\mu_0^2} \left(1 - \frac{2D}{c}(1 - \ln(2))\right)\right]\end{aligned}\quad (7.7)$$

We note that this reduces to the correct form for the strong coupling limit.

Next, we find the energy per unit length to be

$$\begin{aligned}\frac{E}{L} &= \frac{\mu_0 D}{3} \left[1 + \frac{\pi^2 T^2}{4\mu_0^2} g\right] \\ &= \frac{\pi^2 D^3}{3} - \frac{4\pi^2 D^4}{3c}(1 - \ln(2)) + \frac{T^2}{12D} \left(1 + \frac{2D}{c}(1 - \ln(2))\right)\end{aligned}\quad (7.8)$$

which for zero temperature has the same result as equation 5.31, but otherwise extends this result to low temperatures.

Finally, the heat capacity per unit length at constant volume (length) is given by

$$\begin{aligned}\frac{C_v}{L} &= \frac{DTg}{2\mu_0} \left[\frac{\pi^2}{3} - 3\zeta(3)(1 - g)\frac{T}{\mu_0} + \dots\right] \\ &= \frac{T}{6D} \left(1 + \frac{2D}{c}(1 - \ln(2))\right) - \frac{3\zeta(3)T^2}{\pi^4 D^2 c}(1 - \ln(2))\end{aligned}\quad (7.9)$$

In the strong coupling limit, we regain equation 5.11, as required. This shows that unlike the strong coupling limit which had no T^2 dependence, we can expect a T^2 term to appear in the heat capacity. This equation represents a significant enhancement over results previously derived.

Thus, we have presented results from the fractional exclusion statistics formulation for the spinor Bose gas with zero magnetisation. The results agree with previous working, and extend most results beyond what was previously known. In this sense, where non-mutual statistics may be approximated, fractional exclusion statistics provides an excellent means of calculating thermodynamic quantities.

Given the similarity of the form of the statistical interaction g to the charge density ρ , it is possible that similar expressions may be found for arbitrary magnetisations through the use of the Wiener-Hopf method. This would then present an exciting new method of calculating low temperature thermodynamic properties with significant benefits over other methods.

Conclusions and Further Research

8.1 Conclusions

This thesis has presented a comprehensive analysis of the exact Bethe ansatz solution for the spinor Bose gas, starting with the derivation of the Bethe ansatz equations. Two versions of the thermodynamic Bethe ansatz equations were developed, both assuming the string hypothesis and real spin rapidities. The free energy was written as a function of the solution to the thermodynamic Bethe ansatz equations.

The behaviour of the system at the limiting regimes of zero temperature and $c = \infty$ was analysed, where original derivations of density functions for strong and weak coupling regimes with no spin were presented, and the effects of adding spin rapidities to the system were explored. An expansion for the free energy in the strong coupling limit was also presented. The process of fermionisation was discussed, and ground state energies for the spinless and zero magnetisation regimes were derived.

A new approximation was developed which allowed the solution to the thermodynamic Bethe ansatz equations to be found in the strong coupling regime, to order $1/c$. The thermodynamic Bethe ansatz equations in the low temperature approximation suggested by Hatsugai *et al.* were used, and the resulting expression for the free energy was expanded for low temperatures to order T^2 . Using the free energy, the chemical potential and the heat capacity at constant volume were derived as a function of temperature. The significance of these expressions is that a first order correction to the strong coupling limit was obtained, which shows the effect of using spinor particles over spinless particles.

An overview of fractional exclusion statistics was presented, and specific details on Haldane exclusion statistics were given. The ability to calculate thermodynamic quantities for non-mutual statistical interactions from Haldane exclusion statistics was noted. The statistical interaction for the spinless Bose gas first presented by Bernard and Wu was derived, and why attempts to construct an equivalent statistical interaction in multi-component models have failed was discussed. A special case for the spinor Bose gas in which a non-mutual statistical interaction could be approximated was found, and expansions for the energy, heat capacity and chemical potential were derived to higher order than previously possible for that particular system. It was concluded that obtaining a general statistical interaction is only useful for classification purposes and in the rare cases when non-mutual statistics may be obtained.

8.2 Further Research

Throughout this thesis, we have identified many areas that may be further investigated. Here we present some of the more relevant aspects of these areas.

In our derivation of the chemical potential, we approximated its evolution with temperature, and did not obtain M/N dependence. The correct chemical potential could be found by solving the Bethe ansatz equations in integral form, as described in Section 6.7. This solution would also provide the magnetisation at temperature T .

Another important issue that should be addressed is exactly where the $\ln(\phi)$ term in equation 6.44 arises from, and whether or not this contradicts the work of Gu *et al.* If the constant term should be omitted, then the results of Chapter 6 will agree with the limiting cases derived in Chapter 5.

Another step which may be taken is to investigate the Wiener-Hopf method and its solutions for the integral Bethe ansatz equations, in order to find the spinon root and hole density. If solutions similar to those for $M = N/2$ can be found, it may be possible to approximate a non-mutual statistical interaction for the low temperature and strong coupling regime, which would provide a new method for finding thermodynamical information for low temperatures. Furthermore, should the statistical interaction be written in terms of M/N , then the free energy may be minimised over the magnetisation, achieving the magnetisation as a function of temperature.

Bethe ansatz solvable quantum gases are part of a field of active research, which comprises of a number of aspects which were beyond the scope of this thesis. These areas include:

- Magnetic susceptibility, which may be calculated from the free energy when taking into account an external field
- Waves motion. By considering single particle excitations, velocities of both particle waves and spin waves may be calculated, and effective masses for single particles (be they actual particles or pseudoparticles) may be obtained.
- Magnetisation as a function of external field. As discussed at the end of Chapter 6, one can also consider how the magnetisation of the system behaves for different external field strengths at zero temperature. The hysteresis effect may be investigated in this system.
- 3 level systems. Although we have only considered a two level system, it is possible to consider three level systems, for which more conventional (spin-1) bosons may be used experimentally. Li *et al.* have briefly considered the thermodynamics of this system, concentrating mostly on the strong coupling limit. A $1/c$ correction term has not yet been derived for this model.
- Fermions. While we have used bosonic wavefunction symmetries in our model, it is also possible to construct a spinor Fermi gas. The equations for this model are slightly less complicated, but still remain an area of active investigation.

Sommerfeld's Lemma

In Fermi-Dirac statistics, there are a number of integrals of the form

$$F_n(\xi) = \int_0^\infty \frac{x^{n-1} dx}{e^{x-\xi} + 1}$$

where ξ ranges from $-\infty$ to ∞ , and n is usually, although not necessarily, a half-integer. Pathria [44] provides an excellent treatment of integrals of this type in Appendix E, which we have briefly reproduced here.

For ξ large and negative, the integrand may be expanded in powers of $e^{-\xi}$, giving

$$F_n(\xi) \approx \Gamma(n) \left(e^{-\xi} - \frac{e^{-2\xi}}{2^n} + \frac{e^{-3\xi}}{3^n} + \dots \right)$$

where $\Gamma(n)$ is the usual gamma function.

On the other end of the scale, for ξ large and positive, the integral is then dominated by the denominator of the integrand $(e^{x-\xi} + 1)^{-1}$, which has limiting values of zero as $x \rightarrow \infty$, and close to one, as $x \rightarrow 0$. The departure from these limiting values is only of significance in the neighbourhood of $x = \xi$, whose width is of a much smaller order than the order of the entire integral. Thus, for a first order approximation, we can replace the curve with a step function.

Sommerfeld (1928) furthered this approach through an expansion which arrives at Riemann zeta functions, giving the approximation

$$F_n(\xi) \approx \frac{\xi^n}{n} \left(1 + n(n-1) \frac{\pi^2}{6} \frac{1}{\xi^2} + n(n-1)(n-2)(n-3) \frac{7\pi^4}{360} \frac{1}{\xi^4} + \dots \right)$$

This is known as Sommerfeld's Lemma. It is curious to note in the context of this thesis that Sommerfeld was Hans Bethe's Ph.D. supervisor.

Glossary

BAE Bethe Ansatz Equations, see below.

Bethe Ansatz A method for solving one-dimensional problems by using a trial wave-function composed of sums of travelling waves.

Bethe Ansatz Equations A set of non-linear coupled equations, usually the product of numerous fractions, which provide restrictions on the rapidities in the system.

c The strength of the δ -function interaction between particles in the Hamiltonian for the system we are considering. Low values of c corresponding to a weak coupling regime, while high values correspond to a strong coupling regime. At $c = 0$, there is no interaction, and so free bosons are obtained, with an internal degree of freedom. At $c = \infty$, the spin and charge components decouple, leaving effectively free spinless fermions.

Charge Rapidity Also known as a pseudomomentum value, charge rapidities are usually denoted by k_i , and arise as eigenvalues from the Bethe ansatz trial function. They are usually related to the energy and momentum of a system.

Excluson Exclusons are particles which obey Haldane exclusion statistics.

Fermionisation The process by which bosons acquire fermionic properties as the repulsive potential between bosons is increased.

Fractional Exclusion Statistics A form of generalised statistics that allows statistics other than Bose-Einstein and Fermi-Dirac statistics.

g-on gas A gas of particles which have only non-mutual statistics.

Haldane Exclusion Statistics A form of fractional exclusion statistics in which the change in the number of vacant positions of a species is proportional to the change in the number of particles filling all species.

M Total number of particles in the spin- \downarrow state.

Magnetisation The total spin of the system, which is proportional to $N - 2M$.

Mutual Statistics In fractional exclusion statistics, mutual statistics occurs when one species may influence another, and so there is a mutual statistical interaction. Non-mutual statistics occurs when each species only influences particles of the same species, and so we may write $g_{\mu\nu} = \delta_{\mu\nu}g_{\mu}$.

N Total number of particles in the system.

Rapidities Variables in the system which are associated with degrees of freedom, arising as eigenvalues for various equations.

Sommerfeld's Lemma An approximation for a specific form of integral. See Appendix A.

Spin Rapidities Variables that arise as eigenvalues in the derivation of the Bethe ansatz equations. These numbers relate to the energy of spin-wave excitations, and influence a system by altering the scattering processes, and thus the energy of the system.

Spinor Bose Gas A gas of bosons which each have an internal degree of freedom with SU(2) symmetry, i.e., a spin- \uparrow /spin- \downarrow structure.

String Hypothesis The hypothesis that in the thermodynamic limit of a Bethe ansatz solvable model with spin rapidities, the spin rapidities will form "strings" in the complex plane, pairing up to form a bound state.

Strong/Weak Coupling Regime The strong coupling regime is the regime for which the repulsive interaction is strong, or c is large. The weak coupling regime is the regime in which the repulsive interaction is weak, or c is small.

TBA Thermodynamic Bethe Ansatz, see below.

Thermodynamic Bethe Ansatz A method introduced by Yang and Yang [28] for arriving at the equilibrium conditions of a system from the Bethe ansatz equations. Usually referred to as the Thermodynamic Bethe Ansatz equations, or the Thermodynamic Bethe Ansatz method.

Thermodynamic Limit The limit of a system in which N and V are taken to ∞ , but in a ratio such that the particle density N/V is constant. It is assumed that a physical system has such a large number of particles that the probabilistic behaviour becomes statistical, and so the thermodynamic limit of a many-body model describes a physical system.

Bibliography

- [1] http://nobelprize.org/nobel_prizes/physics/laureates/2001/illpres/, viewed 27/10/06
- [2] http://nobelprize.org/nobel_prizes/physics/laureates/1997/illpres/, viewed 27/10/06
- [3] B. Paredes *et al.*, *Tonks-Girardeau gas of ultracold atoms in an optical lattice*, *Nature* **429** 277 (2004)
- [4] L. Pollet, S. M. A. Rombouts and P. J. H. Denteneer, *Ultracold atoms in one-dimensional optical lattices approaching the Tonks-Girardeau regime*, *Phys. Rev. Lett.* **93** 210401 (2004)
- [5] T. Kinoshita, T. Wenger and D. S. Weiss, *Observation of a one-dimensional Tonks-Girardeau gas*, *Science* **305** 1125 (2004)
- [6] E. H. Lieb and W. Liniger, *Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State*, *Phys. Rev.* **130**, 1605 (1963)
- [7] T. Kinoshita, T. Wenger and D. S. Weiss, *Local pair correlations in one-dimensional Bose gases*, *Phys. Rev. Lett.* **95** 190406 (2005)
- [8] H. J. Lewandowski *et al.*, *Observation of anomalous spin-state segregation in a trapped ultracold vapor*, *Phys. Rev. Lett.* **88** 070403 (2002)
- [9] J. M. McGuirk *et al.*, *Spatial resolution of spin waves in an ultracold gas*, *Phys. Rev. Lett.* **89** 090402 (2002)
- [10] L. E. Sadler *et al.*, *Spontaneous symmetry breaking in a quenched ferromagnetic spinor Bose condensate*, arXiv:cond-mat/0605351
- [11] T. L. Ho, *Spinor Bose condensates in optical traps*, *Phys. Rev. Lett.* **81** 742 (1998)
- [12] T. L. Ho and S. K. Yip, *Fragmented and single condensate ground states of spin-1 Bose gas*, *Phys. Rev. Lett.* **84** 4031 (2000)
- [13] E. Eisenberg and E. H. Lieb, *Polarization of interacting bosons with spin*, *Phys. Rev. Lett.* **89** 220403 (2002)
- [14] J. N. Fuchs, D. M. Gangardt, T. Keilmann and G. V. Shlyapnikov, *Spin waves in a one-dimensional spinor Bose gas*, *Phys. Rev. Lett.* **95** 150402 (2005)
- [15] M.T. Batchelor, M. Bortz, X.-W. Guan and N. Oelkers, *Collective dispersion relations for the 1D interacting two-component Bose and Fermi gases*, *J. Stat. Mech.* (2006) P03016
- [16] E. H. Lieb and F. Wu, *Phys. Rev. Lett.* **20**, 1443 (1968)
- [17] R. W. Richardson, *Phys. Lett.* **3**, 277 (1963); **5**, 82 (1963)

- [18] R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964)
- [19] “*Thermodynamics of One-Dimensional Solvable Models*”, by M. Takahashi, Cambridge University Press, United Kingdom, 1999
- [20] “*Beautiful Models*”, by B. Sutherland, World Scientific, Singapore, 2004
- [21] D. Bernard and Y.-S. Wu, *A Note on Statistical Interactions and the Thermodynamic Bethe Ansatz*, arXiv.org:cond-mat/9404025 v2 (1994)
- [22] F. D. M. Haldane, “*Fractional Statistics*” in *Arbitrary Dimensions: A Generalization of the Pauli Principle*, Phys. Rev. Lett **67**, 937 (1991)
- [23] S. B. Isakov, D. P. Arova, J. Myrheim and A. P. Polychronakos, *Thermodynamics for fractional exclusion statistics*, Phys. Lett. A **212**, 299 (1996)
- [24] H. A. Bethe, Z. Phys. **71**, 205 (1931)
- [25] C.-N. Yang, *From the Bethe-Hulthén Hypothesis to the Yang-Baxter Equation*, in Ekspong, G. (ed.): *The Oskar Klein memorial lectures, vol. 1*, 35
- [26] N. Oelkers, M. T. Batchelor, M. Bortz and X.-W. Guan, *Bethe ansatz study of one-dimensional Bose and Fermi gases with periodic and hard wall boundary conditions*, J. Phys. A **39**, 1073 (2006)
- [27] B. Sutherland, *Further Results for the Many-Body Problem in One Dimension*, Phys. Rev. Lett. **20**, 98 (1968)
- [28] C. N. Yang and C. P. Yang, *Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction*, J. Math. Phys. **10**, 1115 (1969)
- [29] S.-J. Gu, Y.-Q. Li, Z.-J. Ying and X.-A. Zhao, *Thermodynamics of Two Component Bosons in One Dimension*, I. J. Mod. Phys. B **16**, 2137 (2002)
- [30] Y. Hatsugai, M. Kohmoto, T. Koma and Y.-S. Wu, *Mutual-exclusion statistics in exactly solvable models in one and higher dimensions at low temperatures*, Phys. Rev. B **54**, 5358 (1996)
- [31] M. Takahashi, *Thermodynamic Bethe Ansatz and Condensed Matter*, arXiv.org:cond-mat/9708087 (1997)
- [32] “*Fractional Statistics and Quantum Theory, 2nd Edition*”, by A. Khare, World Scientific, Singapore, 2005
- [33] C. Nayak and F. Wilczek, *Exclusion Statistics: Low-Temperature Properties, Fluctuations, Duality, and Applications*, Phys. Rev. Lett. **73**, 2740 (1994)
- [34] Y.-S. Wu, *Statistical Distribution for Generalized Ideal Gas of Fractional-Statistics Particles*, Phys. Rev. Lett. **73**, 922 (1994)
- [35] M.T. Batchelor and X.-W. Guan, *Fermionization and fractional statistics in the strongly interacting one-dimensional Bose gas*, Laser Phys. Lett. (2006) DOI 10.1002/lapl.200610068

-
- [36] M. Gaudin, *Boundary Energy of a Bose Gas in One Dimension*, Phys. Rev. A **4**, 386 (1971)
- [37] M. T. Batchelor, X.-W. Guan and J. B. McGuire, *Ground state of 1D bosons with delta interaction: link to the BCS model*, J. Phys. A **37**, 497 (2004)
- [38] Y.-Q. Li, S.-J. Gu, Z.-J. Ying and U. Eckern, *Exact results of the ground state and excitation properties of a two-component interacting Bose system*, Europhys. Lett. **61**, 368 (2003)
- [39] “*Table of Integrals, Series and Products*”, by I. S. Gradshteyn and I. M. Ryzhik, Academic Press, London, 1981
- [40] K. Iguchi, Int. J. Phys. **B11**, 3551 (1997)
- [41] P. Schlottmann, *Low-temperature behaviour of the $S=\frac{1}{2}$ ferromagnetic Heisenberg chain*, Phys. Rev. B **33**, 4880 (1986)
- [42] P. Schlottmann, *Exact Results for Highly Correlated Electron Systems in One Dimension*, Int. J. of Mod. Phys. B **11**, 355 (1997)
- [43] M.T. Batchelor and X.-W. Guan, *Generalized exclusion statistics and degenerate signature of strongly interacting anyons*, arXiv.org:cond-mat/0606353 v2 (2006)
- [44] “*Statistical Mechanics*”, by R. K. Pathria, Pergamon Press, Great Britain, 1980